МАТЕМАТИКА

MATHEMATICS

УДК 517.977.1
ББК 22.19
Ш 96

Шумафов М.М.
Доктор физико-математических наук, профессор, зав. кафедрой математического анализа и методики преподавания математики факультета математики и компьютерных наук Адыгейского государственного университета, Майкоп, тел (8772) 593905, e-mail: magomet_shumaf@mail.ru

Стабилизация неустойчивых состояний равновесия динамических систем*

Часть 2. Стационарная и нестационарная стабилизация, назначение полюсов**

(Рецензирована)

Аннотация. *Дан обзор работ по проблемам стабилизации и размещения собственных значений или полюсов линейных стационарных управляемых систем. Представлены основные результаты, полученные в работах из списка литературы. Приведены результаты исследований по решению проблемы Брокетта о стабилизации линейной управляемой системы нестационарной обратной связью. Сформулированы теоремы низкочастотной и высокочастотной стабилизации линейных систем. В частности, приведены необходимые и достаточные условия стабилизации неустойчивых состояний равновесия двумерных и трехмерных динамических систем в терминах параметров систем. Эти условия показывают, что введение в систему нестационарной обратной связи расширяет в целом возможности обычной стационарной стабилизации. Представленные результаты могут быть использованы при решении задач анализа и синтеза линейных систем управления, а также при исследовании вопросов устойчивости нелинейных управляемых систем в окрестности неустойчивых состояний равновесия.*

Ключевые слова: *линейная управляемая система, неустойчивое состояние равновесия, асимптотическая устойчивость, стабилизация, назначение полюсов, обратная связь по выходу.*

Shumafov M.M.
Doctor of Physics and Mathematics, Professor, Head of Department of Mathematical Analysis and Methodology of Teaching Mathematics of Mathematics and Computer Science Faculty, Adyghe State University, Maikop, ph. (8772) 593905, e-mail: magomet_shumaf@mail.ru

Stabilization of unstable steady states of dynamical systems

Part 2. Stationary and nonstationary stabilization, pole assignment

Abstract*. In this work, the output feedback stabilization and pole assignment problems in the control of linear time-invariant systems are reviewed, and corresponding results are presented along with a literature review. The main results on solving the Brockett's problem on stabilization of linear controllable systems by nonstationary feedback control are presented. The theorems on low- and high frequency stabilization of linear systems are formulated. In particular, necessary and sufficient conditions for stabilization of unstable steady states of two- and three-dimensional dynamical systems in terms of the system parameters are given. These conditions show that an introduction in the system a nonstationary feedback, in general, extends the possibilities of stabilization by ordinary stationary feedback control. The results delivered can be applicable to the analysis and design of linear control systems, and also to the stability analysis of nonlinear control systems in the neighborhood of unstable equilibrium points.*

Keywords*: linear controllable system, unstable steady state, asymptotic stability, stabilization, pole assignment, output feedback.*

1. Introduction

In the first part [1] of the present work a survey on the feedback control stabilization problem of unstable steady states (USSa) of linear controllable dynamical systems was given. The statements of stabilization problems were formulated. Here, in the second part, some approaches to solution of the stabilization and pole assignment problems formulated in [1] and main corresponding results will be presented.

---

* *Работа представляет собой расширенный текст пленарного доклада на Первой Международной научной конференции «Осенние математические чтения в Адыгее», посвященной памяти профессора К.С. Мамия, 8-10 октября 2015 г. Адыгейский государственный университет, Майкоп, Республика Адыгея.*

** *Данная работа является продолжением статьи [1].*

Consider a linear stationary (time-invariant) controllable dynamical system

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t), & \\ y(t) = Cx(t) & \left(\dot{x} \equiv \frac{dx}{dt}\right), \end{cases} \quad \begin{matrix} (1.1) \\ (1.2) \end{matrix}$$

where $x(t)\epsilon\mathbb{R}^n$ is a state vector, $u(t)\epsilon\mathbb{R}^m$ is an input (control) vector, $y(t)\epsilon\mathbb{R}^l$ is an output vector, and *A, B, C* is real constant $(n \times n)$-, $(n \times m)$-, $(l \times n)$-matrices, respectively. The system (1.1), (1.2), can be considered as a linearized system around USS $x = 0$ $(u = 0)$ of nonlinear system $\dot{x} = f(x, u)$, $f(0,0) = 0$.

2. Stationary output feedback stabilization

Recall that the Problem 1 [1] is stated as follows:

Problem 1. *Given a system (1.1), (1.2), find a stationary output feedback*

$$u(t) = Ky(t), \tag{2.1}$$

where K is a real constant $(m \times l)$*-matrix, such that the origin of the closed-loop system*

$$x(t) = (A + BKC)x(t) \tag{2.2}$$

would be asymptotically stable.

More exactly in other words:

Given a triple real matrices (A, B, C). Determine necessary and sufficient conditions on (A, B, C) under which exist a real matrix $K \in \mathbb{R}^{m \times l}$ *such that the all eigenvalues* $\lambda_j(A + BKC)$ $(j = 1, \dots, n)$ *of the matrix A+BKC lie in the open left-half plane:* $\lambda_j(A + BKC) < 0$.

As far as we know the problem above formulated is one of the basic analytically unsolved problems in control theory. There are only partial results obtained in a number of special cases. In the state feedback case, when in (1.2) *C* is the identity $(n \times n)$-matrix, the solution of the Problem 1 is significantly simple and well known.

Theorem 2.1 [2, p. 206]**.** *The system (1.1), (1.2), where C=I, is stabilizable or the pair (A, B) is stabilizable if only if the following condition holds.*

$$rank(\lambda I - A \quad B) = n \qquad \forall \lambda \epsilon \sigma(A) \cap \mathbb{C}^+, \tag{2.3}$$

where I is the $(n \times n)$*-identity matrix,* $\sigma(A)$ *denotes the spectrum of the matrix A, and* $\mathbb{C}^+$ *denotes closed right-half plane:* $\mathbb{C}^+ = \{\lambda \in \mathbb{C}: Re\lambda \geq 0\}$.

Note that the condition (2.3) is weaker than controllability condition of the pair (*A*, *B*) ([3, 4])

$$rank(\lambda I - A \quad B) = n \qquad \forall \lambda \epsilon \sigma(A),$$

which is equivalent to the Kalman controllability condition [5]

$$rank(B \ AB \dots A^{n-1}B) = n.$$

Under controllability condition of the pair (*A*, *B*), an elementary proof of the theorem giving a solution of the stabilization problem in the state feedback case (*C*=*I*) is given in [4–8].

If *B* is the identity matrix, then for stabilizability of system (1.1), (1.2) it is necessary and sufficient that

$$rank(\lambda I - A^T \quad C^T) = n \qquad \forall \lambda \epsilon \sigma(A) \cap \mathbb{C}^+, \tag{2.4}$$

since $\lambda_j(A + KC) = \lambda_j(A^T + C^T K^T)$, where *K* is a control gain matrix in (2.1). (Here T denotes transposition.)

The condition (2.4) is a criterion for stabilizability of the pair (A^T, C^T). In this case the system (1.1), (1.2) or pair (A, C) is said to be *detectable*.

If neither B nor C is the identity matrix, then the stationary static output feedback stabilization problem becomes considerably more difficult.

There are various approaches used to find a stabilizing matrix K in (2.1). We point out some of these ones. In the case of single-input single-output systems (m=1, l=1), graphical approaches (root-focus, Nyquist criterion) are used to answer both the existence and the design questions of stabilizing static output controller (2.1). Also, there exist some necessary and sufficient algebraic tests [9, 10] for the existence of stabilizing output feedbacks. However, these tests are just as complicated as the graphical methods. In addition, they are not easily extendable to the multi-input multi-output case (m+l>2). There are only some specialized cases which may be resolved using the multivariable Nyquist criterion [11]. Below we dwell on some approaches for the solvability of the stabilization and, more general, pole assignment problems by output feedback (2.1).

2.1. Linear-quadratic equation formulation

In [12] necessary and sufficient conditions are given for the stabilizability of the linear system (1.1) using output feedback in terms of the solvability of the special Lur'e-Riccati equation.

Theorem 2.2 (Kučera, Souza [12]). *Suppose that in the system (1.1), (1.2) matrices A, B and C satisfy the following conditions:*

1) the pair (A, B) is stabilizable,

2) the pair (A, C) is detectable.

Then the system (1.1) is stabilizable with static output feedback (2.1) if and only if there exist a real $(m \times n)$*-matrix G such that the coupled linear-quadratic matrix equations on H*

$$KC + B^T H = G, \tag{2.5}$$

$$A^T H + HA - HBB^T H + C^T C + G^T G = 0, \tag{2.6}$$

has a solution $H = H^T$*, where H is positive definite:* $H > 0$.

From (2.5) and (2.6) we have a matrix relation

$$A^T H + HA - HBB^T H + C^T C + (C^T K^T + HB)(KC + B^T H) = 0,$$

which can be rewritten as

$$(A + BKC)^T H + H(A + BKC) = -C^T (I_l + K^T K)C. \tag{2.7}$$

(Here I_l is the identity $(l \times l)$-matrix.)

The left-hand side of relation (2.7) can be considered as a linear operator F defined in the linear space of symmetric matrices $\{H\}, H^T = H$:

$$F(H) = (A + BKC)^T H + H(A + BKC).$$

Since the matrix $(A + BKC)$ is stable (Hurwitzian) by Theorem 2.2, from Lyapunov's lemma [3, p. 61] it follows that the equation $F(H) = M$ is uniquely solvable with respect to H for any matrix $M = M^T$. Denote this solution as $H = \mathcal{L}(M)$, where $\mathcal{L}$ is the inverse operator for F: $\mathcal{L} = F^{-1}$. Then the solution H of equation (2.7) can be represented as

$$H = -\mathcal{L}(C^T C) - \mathcal{L}(C^T K^T K C). \tag{2.8}$$

Substituting (2.8) in the equation (2.5), we obtain a quadratic equation on $(m \times n)$-matrix $h = KC$:

$$B^T \mathcal{L}(h^T h) - h = -G - B^T \mathcal{L}(C^T C). \tag{2.9}$$

In such a form (2.9) the resolving Lure's equations were used in works [13–18] and others. Thus, the equations (2.5), (2.6) are reduced to the Lur'e-Riccati equation.

In [13] and other works the conditions of solvability of Lur'e equations are obtained in the cases m=1, l=2, 3, 4, 5. By using these equations the solutions of many practically important problems were obtained. In the general case the solvability condition of Lure's equations coincides with Yakubovich-Kalman frequency condition [3, 4]. In the linear case, for linear system (1.1), (1.2), it coincides with classical Nyquist criterion.

A related necessary and sufficient conditions of the stabilizability in terms of the solvability of a modified Lur'e-Riccati matrix equation similar to (2.6) was given in the paper [19]. However, one of two assertions formulated in this paper turned out to be inaccurate. In [20] a counterexample is given, and a new theorem correcting the corresponding assertion is given.

2.2. Coupled linear matrix inequality formulation

Now we present a theorem giving another necessary and sufficient conditions for stabilization of system (1.1), (1.2) by output feedback (2.1). These conditions can be obtained in terms of coupled linear matrix inequalities, which follow from a quadratic Lyapunov function approach. Indeed, from Lyapunov stability theory it is well-known that the closed-loop system (2.2) is stable if and only if the matrix K satisfies the following matrix inequality

$$(A + BKC)P + P(A + BKC)^T < 0 \tag{2.10}$$

for some matrix $P > 0$. For a fixed P, the inequality (2.10) is a linear matrix inequality in the matrix K. In [21, 22] necessary and sufficient conditions of output feedback stabilization are obtained by finding the solvability conditions of the inequality (2.10) in terms of K. The following assertion holds.

Theorem 2.3 [21, 22]**.** *For existence of a stabilizing output feedback gain matrix K it is necessary and sufficient the existence of a matrix* $P > 0$ *such that*

$$B^{\perp}(AP + PA^T)(B^{\perp})^T < 0, \tag{2.11}$$

$$(C^T)^{\perp}(A^T P^{-1} + P^{-1}A)[(C^T)^{\perp}]^T < 0, \tag{2.12}$$

where $B^{\perp}$ *and* $(C^T)^{\perp}$ *are full-rank matrices, orthogonal to* B *and* C^T*, respectively* (*i.e.* $B^{\perp}B = 0$, $(C^T)^{\perp}C^T = 0$).

Necessity of the conditions (2.11) and (2.12) is evident. Really, the inequality (2.11) follows from (2.10) by multiplication on the left by matrix $B^{\perp}$ and on the right by $(B^{\perp})^T$. (It follows from that if a quadratic form $x^T Rx < 0 \; \forall x \in \mathbb{R}^n, x \neq 0$ and $Q \in \mathbb{R}^{q \times n}$ is a full-rank matrix, that is, rank $Q = q, q < n$, then $y^T QRQ^T y < 0 \; \forall y \in \mathbb{R}^q, y \neq 0$.) The inequality (2.12) follows from (2.10) by multiplying on the left and right by P^{-1} and then multiplying on the left by $(C^{\perp})^T$ and on the right by $[(C^T)^{\perp}]^T$ (Here $C[(C^T)^{\perp}]^T = 0$ since $(C^T)^{\perp} C^T = 0$.)

In [21, 22] it is shown that the converse is also true, that is, if there exists a matrix $P > 0$ that satisfies the inequalities (2.11) and (2.12) then the inequality (2.10) has a solution K, consequently there exists a stabilizing feedback matrix K. Also, in [21] a parametrization of all static output feedback matrices that correspond to a feasible solution P of inequalities (2.11) and (2.12) is given.

2.3. Other approaches

There are also other methods of solving the stabilizability problem by output feedback. In the paper [23] *nonlinear programming methods* are used in order to solve this problem. The stabilization of the system (1.1) by (2.1) is realized by minimizing the real part of the dominant eigenvalue of the closed-loop system (2.2).

In [24] *decision methods* are suggested to study the output feedback stabilization problem. By using a stability criterion such as Routh-Hurwitz, the stabilizability problem can be reduced to a system of multi-variable polynomial inequalities k_{ij}, which is *ij*-th component of the feedback matrix K in (2.1). Decision methods permit one to establish, in a finite number of algebraic steps, the existence of real variables k_{ij} such that all polynomial inequalities are satisfied. Such methods are currently referred to as *quantifier elimination* or QE techniques (Tarski (1951), Basu et al. (1994)), which uses *Boolean* formulas containing quantified and unquantified real variables. In control problems the unquantified variables are generally the compensator parameters, and the quantified variables are the plant parameters. An important special problem is the QE problem with no unquantified variables, that is, free variables. This problem is referred to as the *general decision problem*, which is stated as to determine if a given quantified formula with no unquantified variables is true or false. The general decision problem may be applied to the problem of existence of compensators in control systems design [25]. According to [25] algorithms for solving general decision problems were first given by Tarski [26] and Seidenberg [27]. These algorithms are commonly called Seidenberg-Tarski decision procedures. Tarski showed that the decision problem is solvable in a finite number of steps, but his algorithm and later modifications are exponential in size of the problem. As is shown in the paper [24] the operations prescribed by Tarski's algorithm for solution of output feedback stabilization problem are tedious, and this made the technique limited.

In the paper [28] methods of algebraic geometry are used for output feedback stabilization of system (1.1), (1.2). The paper [29] focuses on output feedback stabilizability for generic classes of system.

Many of existing approaches for solvability of the stabilization problem use a *dynamic output feedback* (or dynamic compensator) considered as an alternative one to static output feedback (static compensator). It should be pointed out that a main disadvantage of static stationary (time-invariant) output feedback (2.1) is that, its potential is limited in comparison with dynamic output feedback. But static output feedback is important in applications where a feedback is desired to be tuned with a restricted number of parameters. Also, static output feedback requires very few online computations and almost no memory because of that it does not involve state estimation nor the introduction of additional state variables [30].

Note that the case, where a *dynamical* output compensator of arbitrary fixed order $n_c \leq n$ is used, may be brought back to the *static* output feedback case as follows (see e.g. [31, 32]).

Suppose the *dynamic compensator* is given in state space in the form

$$\begin{cases} \dot{x}_c(t) = A_c x_c(t) + B_c y(t), \\ u(t) = C_c x_c(t), \end{cases} \tag{2.13}$$

where $x_c(t) \in \mathbb{R}^{n_c}, A_c \in \mathbb{R}^{n_c \times n_c}, B_c \in \mathbb{R}^{n_c \times l}, C_c \in \mathbb{R}^{m \times n_c}$. Here n_c is a given positive integer denoting the order (i.e. dimension) of the dynamic compensator.

Introducing additional input (control) $u_c(t)$ and output $y_c(t)$ by equalitiesn $u_c(t) = \dot{x}_c(t), y_c(t) = x_c(t)$ and coupling the system (1.1), (1.2) and (2.13) we obtain an augmented state-system

$$\left.\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{pmatrix} &= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x_c(t) \end{pmatrix} + \begin{pmatrix} 0 & B \\ I & 0 \end{pmatrix} \begin{pmatrix} u_c(t) \\ u(t) \end{pmatrix}, \\ \begin{pmatrix} y_c(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x_c(t) \end{pmatrix}, \end{aligned}\right\} \tag{2.14}$$

so that the feedback law is now *static* and is given by

$$\begin{pmatrix} u_c(t) \\ u(t) \end{pmatrix} - \begin{pmatrix} A_c & B_c \\ C_c & 0 \end{pmatrix} \begin{pmatrix} y_c(t) \\ y(t) \end{pmatrix}. \tag{2.15}$$

The system (2.14) and static feedback law (2.15) can be rewritten in a compact form

$$\left.\begin{aligned} &\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t), \\ &\tilde{y}(t) - \tilde{C}\tilde{x}(t), \quad \tilde{u}(t) - \tilde{K}\tilde{y}(t) \end{aligned}\right\}, \tag{2.16}$$

where

$$\tilde{x} = \begin{pmatrix} x \\ x_c \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u_c \\ u \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} y_c \\ y \end{pmatrix},$$

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & B \\ I & 0 \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix}, \quad \tilde{K} = \begin{pmatrix} A_c & B_c \\ C_c & 0 \end{pmatrix}.$$

Note that the problem of stabilization via *dynamic* compensator can be also reduced to stabilization by static output feedbacks in the following way, slightly different from the above. Namely, the coupled systems (1.1), (1.2) and (2.13) may be written as

$$\begin{cases} \dot{x}(t) = Ax(t) + BC_c x_c(t), \\ \dot{x}_c(t) = A_c x_c(t) + B_c C x(t), \end{cases} \quad \text{or} \quad \dot{\tilde{x}}(t) = \begin{pmatrix} A & BC_c \\ B_c C & A_c \end{pmatrix} \tilde{x}(t), \tag{2.17}$$

where the closed-loop system matrix in (2.17) can be presented as

$$\begin{pmatrix} A & BC_c \\ B_c C & A_c \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I_{n_c} \end{pmatrix} A_c \begin{pmatrix} 0 & I_{n_c} \end{pmatrix} + \begin{pmatrix} 0 \\ I_{n_c} \end{pmatrix} B_c \begin{pmatrix} C & 0 \end{pmatrix} + \begin{pmatrix} B & 0 \end{pmatrix} C_c \begin{pmatrix} 0 & I_{n_c} \end{pmatrix}. \tag{2.18}$$

Thus, the closed-loop system (2.17) is presented in a decentralized *static* controller structure of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{r} B_i u_i(t), \; y_i(t) = C_i x(t),$$

with the feedback law: $u_i(t) = K_i y_i(t) (i = 1, \ldots, r)$. In (2.17), (2.18)

$$r = 3, \; K_1 = A_c, \; K_2 = B_c, \; K_3 = C_c.$$

Modern control design techniques provide dynamic output controllers of order equal to the order of the plant ($n_c = n$). In this case necessary and sufficient conditions for stabilizability are well known, namely, the pair (A, B) must be stabilizable and pair (A, C) must be detectable [33]. Dynamic output feedback compensation of fixed order $n_c < n$ is one of important problems in control systems design. Stabilizability problem for the case $n_c < n$ is in general unsolved. Partial results can be found, for instance, in [34, 35].

Drawing conclusions, it should be note that the static output feedback stabilization problem is still open on the whole. The existing necessary and/or sufficient conditions are not efficiently testable. They only succeed in transforming the problem into another unsolved problem or into a numerical problem with no guarantee of convergence to a solution [25]. A common thread throughout these methods is the fact that the problem is equivalent to obtaining the solution of a coupled set of matrix (Lyapunov, Lur'e-Riccati etc.) equations or linear matrix inequalities. For details on output feedback stabilization problem and related questions we refer to surveys [25, 36].

3. Pole assignment with state and output feedbacks

A generalization of the stabilizability problem is pole (i.e. eigenvalue) assignment (placement) problem. In this problem the goal is to determine a feedback gain matrix *K* such that the closed-loop system spectrum would be located within a specified region (for instance, in the left-half complex plane) or placed at specified location in the complex plane. Recall that in the exact terms the pole assignment Problem 2 in [1] is stated in the following way.

Problem 2. *Given a triple real matrices (A, B, C) and an arbitrary set* $\{\mu_j\}_{j=1}^{n}$ *of complex numbers* $\mu_1, \ldots, \mu_n$*, closed under complex conjugation, find an real* $(m \times l)$*-matrix K such that the eigenvalues of the matrix A+BKC are precisely those of the self-conjugate set* $\{\mu_j\}_{j=1}^{n}$*, i.e.*

$$\sigma(A + BKC) = \{\mu_1, \ldots, \mu_n\}. \tag{3.1}$$

(Here σ denotes a spectrum of a matrix.)

We separately consider the full-state-feedback case $u(t) = Kx(t) (C = I)$ and the output-feedback one (2.1) $(C \neq I, \; l < n)$.

3.1. Full state feedback

Here, it is required to select a gain matrix $K \epsilon \mathbb{R}^{m \times n}$ in the feedback $u(t) = Kx(t)$ such that the relation (3.1), where $C = I$, would be valid, i.e.

$$\sigma(A + BK) - \{\mu_1, \dots, \mu_n\}. \tag{3.2}$$

As is well known this problem was stated and solved by V.I. Zubov [37] and W.M. Wonham [38].

Theorem 3.1 (Zubov-Wonham [37, 38])**.** *For existing a real* $(m \times l)$*-matrix K such that the relation (3.2) be valid it is necessary and sufficient that the system (1.1) or the pair (A, B) be completely controllable, i.e.* $rank(B\ AB\ \dots A^{n-1}B) = n$.

According to Kalman et al. [39] the Theorem 3.1 was first obtained for the single-input case (*m*=1) by J.E. Bertram in 1959 using root locus method. The same result in terms of linear algebra was formulated and proved (but not published) by R.W. Bass in 1961. The single-input case was also considered by Rissanon [40] and Rosenbrock [41]. In [42, 43] Popov rendered a method of construction of elements of the matrix *K* in multi-input case (*m*>1) where the elements of the parameter matrices *A* and *B* may be arbitrary complex numbers. Also, in multi-input case where $\dot{u}(t) = Kx(t) + Gu(t)$, and where the elements of the matrices *A*, *B* and *K* may be complex numbers a theorem similar to the Theorem 3.1 was proved by Langenhop [44]. Zubov and Wonham were the first to extend the previous results concerning the problem (3.2) from single-input to multi-input systems of the form (1.1) for *real* matrices *A*, *B* and *K*. Other contributions concerning pole assignment in multi-input systems in real case by full state feedback are due to Chen [45], Davison [46], Heymann [47], Simon and Mitter [48], Brunovsky [49], Aksenov [50], Yakubovich [51], Smirnov [52], Leonov and Shumafov [4, 8, 53, 54].

Since then when Zubov's and Wonham's works [37, 38] appeared, there have been written great number (literally hundreds) of papers concerning pole assignment and its applications.

It should be noted that the proof of Zubov's and Wonham's theorem in multi-input and real case is rather tedious. Therefore, after publication of works [37] and [38] there were offered alternative proofs of the Theorem 3.1 to simplify its proof (see, for instance, [45–52]). A simple proof of the theorem was proposed by Aksenov [50] in the case where matrices *A*, *B* and *K* are complex. Notice that the proof of Theorem 3.1 in complex case is far simpler than real one. In [4, 8, 53, 54] a new and the simplest proof of Zubov's and Wonham's theorem presented. This proof 1) both for scalar (*m*=1) and vector (*m*>1) cases gives a uniform algorithm of solving the pole assignment problem (3.2), and 2) only makes use of the well-known fact of reducing a matrix to Jordan normal form (in fact, to a triangle form). This algorithm is convenient and effective for numerical computation of the gain matrix *K* in feedback (2.1).

3.2. Output feedback

From a practical standpoint when considering large order systems and the cost of measuring and feeding back variables it is more desirable a procedure based upon feeding back only the measured variables, i.e. *output feedback*. Hence, we have the motivation for the Problem 2.

Among the first to respond to the Problem 2 was Davison [55]. He proved the following statement.

Theorem 3.2 (Davison [55])**.** *Suppose the system (1.1) is controllable, and* $rank\, C - l \le n$. *Assume that the matrix A is cyclic. Then there exist a real matrix K such that l eigenvalues of the closed-loop system matrix A+BKC are arbitrary close (but not necessary equal) to the l preassigned arbitrary complex numbers, which are closed under complex conjugation.*

Recall that a matrix *A* is said to be cyclic, if the matrix $\lambda I - A$ has only one non-unity invariant polynomial or just the same in Jordan normal form of the matrix A to different boxes correspond different eigenvalues. In [56] it is shown that if the system (1.1), (1.2) is controllable and observable (the pair (A, B) is controllable, the pair (A, C) is observable) then almost any *K* will yield a cyclic matrix $\bar{A} = A + BKC$. Moreover for almost any choice of a vector *q* the pair $(\bar{A}, Bq)$ will be controllable. In [57] this approach was exploited to prove a theorem which strengthens the Theorem 3.2.

Theorem 3.3 (Davison, Chatterjee [57]). *Suppose the pair (A, B) is controllable and the pair (A, C) is observable, and* $rank\, B = n \le n, rank\, C = l \le n$. *Then there exists a real matrix K such that* $max\,(l, m)$ *eigenvalues of the closed-loop system matrix A+BKC are arbitrary close (but not necessary equal) to max (l, m) preassigned arbitrary complex numbers closed under complex conjugation.*

In the case when the matrix *A* is cyclic an alternative proof of the Theorem 3.3 based on another approach was suggested by Sridhar and Lindorff [58]. An analogous result under different conditions is established in Jameson's paper [59] for the systems with scalar input (*m*=1). In later papers Davison and Wang [60], and Kimura [61, 62] proved the following theorem.

Theorem 3.4 (Davison & Wang [60], Kimura [61, 62]). *Suppose the pair (A, B) is controllable and the pair (A, C) is observable with B and C of full rank, i.e.* $rank\, B = m < n, rank\, C = l < n$. *Then for almost all A, B and C there exists an output gain real matrix K such that the matrix A+BKC has* $min\,(n, m + l - 1)$ *eigenvalues arbitrary close to* $min\,(n, m + l - 1)$ *preassigned arbitrary complex numbers closed under complex conjugation.*

Here the words "for almost all" mean that if a certain property P is a function in a matrix *X*, then the set of matrices $\{X_\alpha\}$, for which the property P is not true, is either an empty set or the subset of zero-sets(hypersurfaces) of a finite number of some polynomials $f_i(x_1, \ldots, x_r)(i = 1, \ldots k)$ in elements $x_1, \ldots, x_r$ of the matrix *X*. In this case it is said that the property P is generic.

From Theorem 3.4 it follows *sufficient conditions* for generic "arbitrary close" pole assignability, where the relation (3.1) is approximately fulfilled with any accuracy.

Corollary. If $m + l \ge n + 1$ then the "arbitrary close" pole assignment problem (3.1) is solvable for almost all *A*, *B* and *C*.

An alternative proof of this was offered by Brockett and Byrnes [11], and Shumacher [63]. To formulate one of results established in [11, 63], we recall notions of the rank and the McMillan degree of dynamical system. It is known [64] that the dynamical system (1.1), (1.2) admits a kernel representation

$$P(D)w = 0, \quad w \in C^\infty(\mathbb{R}, \mathbb{R}^{m+l}), \tag{*}$$

where *P* is a polynomial matrix. There are two important invariants of system (*), which are rank *r* and the McMillan degree *n*. The *rank* of the polynomial *P* is called rank of the system (*). A representation (*) is called *row minimal* if the matrix *P* has full row rank. The *McMillan degree n* of the system (*) is defined as the maximal degree of the full size minors in one (and therefore any) minimal representation.

Theorem 3.5 (Brockett & Byrnes [11], Shumacher [63]). *If ml=n, and the number*

$$d(m, l) = \frac{1!2!\ldots(l-1)!(ml)!}{m!(m+1)!\ldots(m+l-1)!}$$

is odd, then the generic rank l system of McMillan degree n is arbitrary pole assignable by static real compensators.

The criterion of oddness $d(m, l)$ is established by Bernstein [65].

Proposition (Berstein [65]). *The number* $d(m, l)$ *is odd if and only if* $min(m, l) = 1$ *or* $min(m, l) = 2$ *and* $max(m, l) = 2^k - 1$, *where k is a positive integer.*

A sufficient condition, when $d(m,l)$ is even, was obtained by Wang [66].

Theorem 3.6 (Wang [66]). *If $d(m,l)$ is even and $ml > n$, then the generic rank l system of McMillan degree n is arbitrary pole assignable by real static feedback compensators.*

The general necessary condition for generic pole assignability was established by Willems and Hesselink [67].

Theorem 3.7 (Willems & Hesselink [67]). *The necessary condition for generic pole assignability by real static output feedback (2.1) is the inequality $ml \geq n$.*

The following proposition states that in general necessary condition $ml \geq n$ is not a sufficient condition for generic pole assignment with *real* static output compensators.

Proposition (Willems & Hesselink [67]). *If $m = l = 2$ and $n = 4$, then the static pole assignment problem is not generically solvable by real output feedback (2.1).*

Remark. Note that in many works (see, for instance, [68, 69], survey [25], and bibliography in [8]) it is also considered a more general, than pole assignment, the *eigenstructure assignment problem*, in which the eigenvalues of the closed-loop system matrix together with the corresponding eigenvectors or invariant factors or elementary divisors are preassigned.

4. Nonstationary stabilization. Brockett's problem

In 1999 R. Brockett [70] formulated a problem of the stabilizability of a stationary (time-invariant) linear system by means of a static nonstationary (time-varying) linear output feedback. To solve this problem two approaches are developed. The first of them is developed for constructing a low-frequency non-stationary feedback, and the second approach for constructing a high-frequency one. The Brockett's problem is formulated as follows (Problem 3 [1]).

Brockett's problem. *Given a linear system (1.1), find a static nonstationary output feedback*

$$u(t) = K(t)y(t) \tag{4.1}$$

such that the closed-loop system

$$\dot{x}(t) = (A + BK(t)C)x(t), \qquad x(t) \in \mathbb{R}^n, \tag{4.2}$$

would be asymptotically stable.

A linear system is called asymptotically stable if all its solutions are asymptotically stable. The latter as well-known is equivalent to asymptotical stability of the trivial solution $x(t) \equiv 0$. Therefore we will say with respect to linear systems about its asymptotic stability.

In the previous section the stationary stabilization by the feedback (4.1) with a constant matrix $K(t) \equiv K$ is considered. In the Brockett's problem it is required to find a variable stabilizing matrix $K(t)$. From this point of view, the Brockett problem can be reformulated as follows.

How much would the use of matrices depending on time t extend the possibilities of classical stationary stabilization?

The solution of the Brockett problem in the class of piecewise constant periodic matrix-functions $K(t)$ with a sufficiently large period (low-frequency stabilization) is given by G.A. Leonov [4, 8, 71–74].

For single-input single-output system $(m = l = 1)$ the Brockett problem is solved by L. Moreau and D. Aeyels [30, 75, 76] in other class of the stabilizing functions, a namely in the class of continuous periodic functions with a sufficiently small period (high-frequency stabilization).

4.1. Nonstationary low-frequency stabilization

Below we will formulate the main results obtained by Leonov [71–74] in solving the Brockett problem.

4.1.1. Multi-input multi-output case $(m \geq 1, l \geq 1)$

Suppose that there exist real constant matrices K_1 and K_2 such that the linear system

$$\dot{x}(t) = (A + BK_jC)x(t), \quad x(t) \in \mathbb{R}^n \quad (j = 1,2), \tag{4.3}$$

possess stable invariant linear manifolds L_j, and invariant linear manifolds M_j such that

$$M_j \cap L_j = \{0\}, \ \dim L_j + \dim M_j = n. \tag{4.4}$$

(The sign "dim" denotes dimension of linear space.)

Assume also that for a solution $x_j(t, x_0), x_j(0, x_0) = x_0$, of the system (4.3) there exist positive numbers $\lambda_j, \mu_j, \alpha_j, \beta_j$ such that the following inequalities

$$\|x_j(t, x_0)\| \leq \alpha_j \|x_0\| \exp(-\lambda_j t) \quad \forall x_0 \in L_j, \tag{4.5}$$

$$\|x_j(t, x_0)\| \leq \beta_j \|x_0\| \exp(\mu_j t) \quad \forall x_0 \in M_j \tag{4.6}$$

are satisfied.

Further suppose that there exists a continuous matrix function $S(t)$ and a number τ such that the transformation g_0^τ in the time from $t = 0$ to $t = \tau$ of the system

$$\dot{x}(t) = (A + BS(t)C)x(t) \tag{4.7}$$

takes the manifold M_1 to a manifold lying in L_2: $g_0^\tau M_1 \subset L_2$. Note that the matrix of transformation g_0^τ is the fundamental matrix $X(t), X(0) = I$, of the system (4.7).

Under these main assumptions the following fundamental theorem is true.

Theorem 4.1 (Leonov [72–74]). *If the inequality $\lambda_1\lambda_2 > \mu_1\mu_2$ holds, then there exists a piecewise periodic matrix-valued function $K(t)$ such that the system (4.2) is asymptotically stable.*

For two-dimensional case from Theorem 4.1 it follows the following statement.

Theorem 4.2 (Leonov [72–74]). *Let n=2. Suppose that there exist matrices K_0 and S_0 satisfying the following conditions:*

1) $\det BK_0C \neq 0$, $Tr BK_0C \neq 0$; if $\det BK_0C = 0$, then at least one of the inequalities $\det A \neq 0$ or $\det(a_1, r_2) + \det(r_1, a_2) \neq 0$ is valid, where a_1, a_2 and r_1, r_2 are the first and the second columns of the matrices A and BK_0C, respectively;

2) the matrix $A + BK_0C$ has complex-conjugate eigenvalues.

Then there exists a piecewise constant periodic matrix-valued function $K(t)$ such that the system (4.1) is asymptotically stable. In this case the periodic matrix function $K(t)$ can be chosen as

$$K(t) = \begin{cases} \gamma K_0 & \text{for } t \in [0, t_1), \\ S_0 & \text{for } t \in [t_1, t_1 + \tau), \\ \gamma K_0 & \text{for } t \in [t_1 + \tau, t_1 + t_2 + \tau), \end{cases} \qquad K(t + T) = K(t), \tag{4.8}$$

where γ is a sufficiently large number, and $T = t_1 + t_2 + \tau$. Here t_1 and t_2 are sufficiently large positive numbers.

4.1.2. Single-input single-output case $(m = l = 1)$

Now we consider the important in practice a special case, a namely, single-input single-output one. In this case B is a column matrix, C is a row matrix: $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$. Suppose that the system (1.1), (1.2) is controllable and observable. The following theorem is corollary of the fundamental Theorem 4.1.

Theorem 4.3 (Leonov [72]). *Suppose that*

$$B\in\mathbb{R}^{n\times 1},\ C\in\mathbb{R}^{1\times n}, \dim M_1 = 1, \dim L_2 = n-1,$$

and the condition $\lambda_1\lambda_2 > \mu_1\mu_2$ *of Theorem 4.1 is fulfilled. Also, assume that there exists a number* S_0 *such that* $S_0 \neq K_j$ $(j = 1,2)$, *where* K_1 *and* K_2 *are number from (4.3) and the matrix* $A + S_0BC$ *has two complex eigenvalues* $\lambda_0 \pm i\omega_0$, *and the remaining eigenvalues* λ_k *satisfy the condition* $Re\,\lambda_k < \lambda_0 (k = 1,\dots,n-2)$.

Then there exist a piecewise constant periodic function (step function) $K(t)$ *such that the system (4.2) is asymptotically stable.*

In this case the periodic function $K(t)$ *is of the form*

$$K(t) = \begin{cases} K_1 & \text{for } t\in[0,t_1), \\ S_0 & \text{for } t\in[t_1,t_1+\tau), \\ K_2 & \text{for } t\in[t_1+\tau,t_1+t_2+\tau), \end{cases} \qquad K(t+T) = K(t), \tag{4.9}$$

where $T = t_1 + t_2 + \tau$. *Here* t_1 *and* t_2 *are sufficient large positive numbers.*

Below we formulate other theorems concerning the scalar case. We consider separately the cases 1) $CB \neq 0$, 2) $CB=0, \dim L_2=n-1$ and 3) $CB = 0, \dim L_2 = n-2$.

Theorem 4.4 (Leonov [72]). *Let* $B\in\mathbb{R}^{n\times 1}$, $C\in\mathbb{R}^{1\times n}$. *Suppose that* $CB \neq 0$ *and the matrix* A *has a positive eigenvalue* μ *and* $n-1$ *eigenvalues with real parts less than* $-\lambda$, *where* $\lambda > \mu$. *Assume that* $\dfrac{CB}{\lim_{p\to\mu}(\mu-p)W(p)} < 1$, *where* $W(p) = C(Ip-A)^{-1}B$ *is a transfer function of the system (1.1), (1.2).*

Then there exists a piecewise constant periodic function $K(t)$ *of the form (4.9) such that system (4.2) is asymptotically stable.*

Theorem 4.5 (Leonov [72]). *Let* $B\in\mathbb{R}^{n\times 1}$, $C\in\mathbb{R}^{1\times n}$. *Suppose that*

$$CB = 0,\ \dim M_1 = 1,\ \dim L_2 = n-1,$$

and the inequality $\lambda_1\lambda_2 > \mu_1\mu_2$ *from Theorem 4.1 holds.*

Then there exists a piecewise constant periodic function $K(t)$ *of the form (4.9) such that system (4.2) is asymptotically stable.*

Theorem 4.6 (Leonov [72]). *Let* $B\in\mathbb{R}^{n\times 1}$, $C\in\mathbb{R}^{1\times n}$. *Suppose that*

$$CB = 0,\ \dim M_1 = 1,\ \dim L_2 = n-2,$$

and the inequality $\lambda_1\lambda_2 > \mu_1\mu_2$ *is true. If the assumptions of the Theorem 4.3 are fulfilled for some number* $S_0 \neq K_j$ $(j = 1,2)$, *then there exists a piecewise constant periodic function* $K(t)$ *of the form (4.9) such that system (4.2) is asymptotically stable.*

Above formulated Theorems 4.1–4.6 yield sufficient conditions for stabilization of the system (1.1), (1.2) by nonstationary feedback (4.1). Now we turn to conditions, which are necessary for stabilization of system (1.1), (1.2).

4.1.3. Necessary stabilization conditions

We consider the scalar case $(m = l = 1)$, which is important for control theory. Assume that the system (1.1), (1.2) is controllable and observable. As is well known the latter is equivalent to non-degeneracy of transfer function. The system (1.1), (1.2) can be written in the following canonical form (see [2, 33, 77]):

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \vdots \\ \dot{x}_{n-1}(t) = x_n(t), \\ \dot{x}_n(t) = -a_1x_1(t) - \dots - a_nx_n(t) + u(t), \ y(t) = c_1x(t) + \dots + c_nx_n(t), \end{cases} \tag{4.10}$$

where the numbers $a_j (j = 1, \dots, n)$ are the coefficients of the characteristic polynomial of the matrix A in (1.1):

$$\det(pI - A) = p^n + a_np^{n-1} + \dots + a_2p + a_1,$$

and c_j are some numbers.

In what follows we assume that $c_n \neq 0$. In this case without loss of generality we may put $c_n = 1$. The following theorem yields necessary condition for stabilization of the system (4.10) by feedback (4.1).

Theorem 4.7 (Leonov [72]). *Suppose that for system (4.10) the following conditions are satisfied:*

1) for $n > 2 \quad c_1 \leq 0, \dots, c_{n-1} \leq 0 \ \ (for\, n = 2 \ \ c_1 \leq 0)$,

2) $c_1(a_n - c_{n-1}) > a_1, \qquad c_1 + c_2(a_n - c_{n-1}) > a_2,$

$c_{n-2} + c_{n-1}(a_n - c_{n-1}) > a_{n-1}.$

Then there is no function $K(t)$ *for which the system (4.10), and therefore the system (1.1), (1.2), is asymptotically stable.*

Another condition ensuring instability of the system (4.2) is well known.

Proposition [78]. *The system (4.2) is unstable if* $Tr\,(A + BK(t)C) \geq \alpha > 0$ *for all* $t\epsilon\mathbb{R}$ *and is not asymptotically stable if* $Tr\,(A + BK(t)C) \geq 0$ *for all* $t\epsilon\mathbb{R}$.

In the following subsection necessary and sufficient conditions for low-frequency stabilizability of two- and three-dimensional systems are presented.

4.1.4. Low-frequency stabilization of two- and three-dimensional systems

Now we apply the above formulated theorems to the cases where $n = 2$ and $n = 3$. We will consider single-input single-output systems $(m = l = 1)$. In this case *B* is a column vector, *C* is a row vector, and $K(t)$ is a scalar function.

1. Consider a second-order linear system with single- input and single- output written in the canonical form (see (4.10))

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -a_1x_1(t) - a_2x_2(t) + u(t), \ \ y(t) = c_1x_1(t) + c_2x_2(t), \end{cases} \tag{4.11}$$

where a_1, a_2, c_1 and c_2 are some real numbers.

Without loss of generality we assume that $c_2 = 1$. Suppose that system (4.11) is controllable and observable, i.e. $c_1^2 - a_2c_1 + a_1 \neq 0$. By applying Theorem 4.2 or Theorem 4.4 and Theorem 4.7 to system (4.11) we can obtain the following statement.

Theorem 4.8 [72]. *Suppose that the inequality* $c_1^2 - a_2c_1 + a_1 \neq 0$ *holds. Then the system (4.11) is stabilizable by feedback (4.1) if and only if at least one of the conditions holds*

$$1)\ c_1 > 0 \quad or \quad 2)\ c_1 \leq 0,\ c_1^2 - a_2c_1 + a_1 > 0. \tag{4.12}$$

In this case a stabilizing control $u(t) = K(t)y(t)$ *can be chosen such that the function*

$K(t)$ is piecewise constant periodic one of the form (4.8), where K_0, $S_0 \in \mathbb{R}$, with sufficiently large period (low-frequency stabilization).

From Routh-Hurwitz conditions it follows that the *stationary* stabilization of system (4.11) by feedback $u(t) = Ky(t)$ $(K = const)$ is possible if and only if either $c_1 > 0$ or the inequalities $c_1 \leq 0$, $a_2 c_1 < a_1$ hold. As we can see the conditions (4.12) define a wider stabilizability domain in the parameter space of the system (4.1) $(c_2 := 1)$ than the domain defined by Routh-Hurwitz conditions for stationary stabilization.

2. Consider a third-order linear system

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -a_1 x_1(t) - a_2 x_2(t) - a_3 x_3(t) + u(t), \ \ y(t) = x_1(t), \end{cases} \tag{4.13}$$

where a_1, a_2, a_3 are some real numbers.

Applying Theorem 4.6 and Theorem 4.7 to the system (4.13) we can obtain the following result.

Theorem 4.9 [72]. *For the system (4.13) to be stabilizable by means of output feedback (4.1) it is necessary and sufficient $a_3 > 0$. In this case a stabilizing function $K(t)$ in the feedback (4.1) can be chosen as piecewise constant periodic function of the form (4.9) with sufficiently large period.*

The Routh-Hurwitz conditions yield that the *stationary* stabilization of system (4.13) by feedback $u(t) = Ky(t)$ $(K = const)$ is possible if and only if $a_2 > 0, a_3 > 0$. As is obvious like Theorem 4.8, the Theorem 4.9 illustrates advantages of nonstationary stabilization in comparison with stationary one.

3. Consider a linear system

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -a_1 x_1(t) - a_2 x_2(t) - a_3 x_3(t) + u(t), \ \ y(t) = x_2(t), \end{cases} \tag{4.14}$$

where a_1, a_2, a_3 are real numbers. By Theorem 4.5 and Theorem 4.7 it can be obtained the following assertion.

Theorem 4.10 [72]. *Suppose that $a_1 \neq 0$, $a_3 \neq 0$. Then the system (4.14) is stabilizable by feedback (4.1) if and only if $a_3 > 0$. In this case a stabilization function $K(t)$ can be chosen as piecewise constant periodic function of the form (4.9) with sufficiently large period.*

By Routh-Hurwitz conditions the *stationary* stabilization of the system (4.14) is possible if and only if $a_1 > 0, a_3 > 0$. Here it is seen just as well the advantages of nonstationary stabilization with comparison with stationary one.

4. Consider a linear system

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \dot{x}_3(t) = -a_1 x_1(t) - a_2 x_2(t) - a_3 x_3(t) + u(t), \ \ y(t) = x_3(t), \end{cases} \tag{4.15}$$

where a_1, a_2, a_3 are some real numbers.

With the help of fundamental Theorem 4.1 and a special construction of stabilizing function $K(t)$, taking into account Theorem 4.7, it can be proved the following statement.

Theorem 4.11 [72]. *Suppose that $a_1 \neq 0, a_2 \neq 0$. For the system (4.15) to be stabilizable it is necessary and sufficient $a_2 > 0$.*

The *stationary* stabilization of system (4.15) is possible if and only if $a_1 > 0, a_2 > 0$. As we can see Theorem 4.11 nicely illustrates advantages of nonstationary stabilization compared to stationary one.

Thus we can conclude, that the use of time-varying output feedback (4.1) extends the possibilities of stationary stabilization of system (1.1), (1.2) by time-invariant output feedback $u(t) = Hy(t)$ $(K = const)$.

4.2. Nonstationary high-frequency stabilization

Here we consider another approach for solving the Brockett problem. This approach is proposed by L. Moreau and D. Aeyels [30, 75, 76], and based on the averaging method. Also this approach is related with techniques from vibrational control theory [79, 80] and well-known phenomenon that the upper equilibrium position of a pendulum becomes stable if the point of suspension performs sufficiently fast oscillations in the vertical direction [81–83]. Below we present main results of the paper [30]. In this paper some *sufficient* conditions are derived for single-input single-output system of the form (1.1), (1.2) $(m = l = 1)$. Feedback gains of the form

$$K(t) = \alpha + \beta\omega^k \cos(\omega t), \quad \alpha, \beta \in \mathbb{R},$$

where k is a natural number, are proposed in [30] in order to stabilize the system (4.1), (4.2) by feedback (4.1). For this purpose it is assumed that the parameter ω is large, which equivalent to that the feedback gain is fast time varying with large amplitude.

4.2.1. High-frequency stabilization theorems

We formulate two theorems which present results of the paper [39]. The first result is concerned with the generic case $CB \neq 0$, the second result with the degenerate case $CB = CAB = \cdots = CA^{2k-1} = 0$.

Theorem 4.12 (Moreau, Aeyels [30]). *Let B be a column vector and C be a row vector in the system (1.1), (1.2). Suppose that $CB \neq 0$. Assume that there exist real numbers λ and $\mu \geq 0$ such that the eigenvalues of the matrix*

$$A + \lambda BC + \mu(CB)BCA$$

are located in the open left half-plane. Then there exists a periodic function

$$K(t) = \alpha + \beta\omega \cos(\omega t), \tag{4.16}$$

where

a) $\alpha = \lambda + \mu(CAB)$,

b) β *is determined by* $\left(\frac{1}{2\pi}\int_0^{2\pi} exp(\beta CB \sin t)dt\right)^2 = \mu(CB)^2 + 1$, *and*

c) $\omega > 0$ *is sufficiently large number, such that the output feedback (4.1) uniformly exponentially stabilizes the system (1.1), (1.2).*

It can be shown that there exists a unique (up to a minus sign) solution β of equation from b) of Theorem 4.12 for all nonnegative values of μ.

Theorem 4.13 (Moreau, Aeyels [30]). *Let B be a column vector and C be a row vector. Suppose that $CB = CAB = \cdots = CA^{2k-1}B = 0$ $(k \in \mathbb{N})$. Assume that there exist real numbers λ and $\mu \geq 0$ such that the eigenvalues of the matrix*

$$A + \lambda BC + (-1)^k(2k+1)\mu(CA^{2k}B)BCA$$

are located in the open left half-plane. Then there exists a periodic function

$$K(t) = \alpha + \beta\omega^{k+1} \cos(\omega t), \tag{4.17}$$

where

a) $\alpha = \lambda + (-1)^k \mu(CA^{2k+1}B)$,

b) $\beta = \sqrt{2\mu}$ *and*

c) $\omega > 0$ *is sufficiently large, such that the output (4.1) uniformly exponentially stabilizes the system (1.1), (1.2).*

For some low-order systems, the numbers λ and μ can be determined analytically [75, 76]. Notice that the stabilizing effect of the proposed feedback laws is guaranteed only for *sufficiently large* ω, but no explicit information is given on how large ω should be taken. Explicit bounds on ω may be obtained from theoretical considerations. We turn our attention to averaging.

Consider the linear differential equation

$$\dot{x}(t) = A(\omega t)x(t), \tag{A}$$

where $A(\tau)$ is a continuous matrix-valued periodic function with period $T > 0: A(\tau + T) = A(\tau)$, and $\omega > 0$ is a constant parameter. Assume that the parameter ω is large. The associated to (A) average system is the equation

$$\dot{x}(t) = A_0 x(t), \tag{B}$$

where A_0 is defined by $A_0 - \frac{1}{T}\int_0^T A(\tau)d\tau$. The stability properties of the "fast time-varying" system (A) and its average (B) are related by a classical result from averaging theory.

Proposition (Averaging and stability [30])**.** *If the origin of system (B) is exponentially stable, then there is a number* $\omega^* > 0$ *such that the origin of system (A) is uniformly exponentially stable for all* $\omega > \omega^*$.

It is possible to give an explicit upper bound for ω^*, but this theoretical bound will typically be very conservative [30]. With this conservative value the feedback laws (4.1), where the gain $K(t)$ is defined from (4.16) or (4.17), will typically be fast-varying with large amplitude. For some applications this may be an undesirable feature. Therefore it is desirable to determine a suitable, less conservative value for ω using numerical simulations.

4.2.2. High-frequency stabilization of two- and three-dimensional systems

Applying Theorem 4.12 and Theorem 4.13 to second-order system (4.11) and third-order system (4.13) it can be obtained sufficient conditions of high-frequency stabilizability of these systems by static output feedback (4.1) with continuous periodic function $K(t)$ of the forms (4.16) and (4.17), where parameter ω is sufficiently large (the period is sufficiently small). It turn out that these conditions coincide with the necessary and sufficient conditions of low-frequency stabilizability of systems (4.11) and (4.13) yielding by Theorem 4.8 and Theorem 4.9.

Compared with static stationary output feedback (2.1), the two- and three-dimensional systems considered very well illustrates the additional possibilities opened up by introducing time variance in the feedback gain. Thus, we can draw a conclusion that the use of nonstationary stabilization has advantages in comparison with stationary stabilization.

We note that there are also many works devoted to stabilization and pole assignment problems for discrete-time systems. For such systems some stabilizability and pole assignability results including solution of discrete analogous of the Brockett problem can be found, for instance, in book [8, ch. 6].

5. Further issues

The main results in the area of stabilization and pole assignment for linear systems are presented in a great number of publications including papers, surveys, and books (see, for instance, surveys [25, 36, 69, 84–87] and books [2, 3, 4, 8, 33, 37, 39, 52, 77, 88]). Nevertheless the problems of stabilization and pole assignment, and also related questions for linear systems remain to be the focus of attention of many scholars and researches. The interest in these problems is motivated first of all by needs of control design, applied engineering and in response to the practical problems of celestial mechanics. The flow of publications in this area of control theory continues to be intensive. We dwell on some late results obtained in these publications.

In the paper [89] a necessary and sufficient condition for modal controllability of a linear differential equation by an incomplete (output) feedback is obtained. In [90–92] for linear and bilinear stationary control systems closed by an output feedback necessary and sufficient conditions for solvability of the pole assignment problem are presented in the case of system coefficients of the special form.The paper [93] presents an algorithm for solving generalized static output feedback pole assignment problem of the following form:

Given matrices $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $C\in\mathbb{R}^{l\times n}$ *and* n *closed subsets* $S_1,\dots,S_n$ *of the complex plane C, find a static output feedback* $K\in\mathbb{R}^{m\times l}$ *that places in each of these subsets a pole of the closed-loop system, i.e.* $\lambda_j(A+BKC)\in S_j$ *for* $j=1,\dots,n$.

This problem encompasses many types of pole assignment problems. For example:

a) classical pole assignment: $S_j=\{\mu_j\}$, $\mu_j\in\mathbb{C}$;

b) stabilization type problems for continuous- time and discrete- time systems:

$S_j=\{z\in\mathbb{C}: Re\, z<0\}$ and $S_j=\{z\in\mathbb{C}: |z|<1\}$ $(j\in\{1,\dots,n\})$, respectively;

c) relaxed classical pole assignment:

$$S_j=\{z\in\mathbb{C}: |z-\mu_j|\le r_j\},\ \ j\in\{1,\dots,n\}.$$

Here each region S_j is a disk centered at $\mu_j\in\mathbb{C}$ with radius $r_j\ge 0$.

The algorithm presented is iterative and is based on alternating projection ideas. Each iteration of the algorithm involves a Schur matrix decomposition and a standard least-squares problem. Also computational results are presented to demonstrate the effectiveness of the algorithm.

In [94] a stabilization criterion for matrices is given. The problem considered is stated as follows:

Given an unstable matrix $A\in\mathbb{R}^{n\times n}$ *and a matrix* $D\in\mathbb{R}^{m\times m}$, *when do there exist matrices* $B\in\mathbb{R}^{n\times m}$ *and* $C\in\mathbb{R}^{m\times n}$ *such that the matrix* $M:=\begin{pmatrix} A & B\\ C & D\end{pmatrix}$ *is stable?*

In other words, when can A be stabilized with a dilation? When this is the case a linear unstable system $\dot{x}(t)=Ax(t)$ can be dilated to a stable system of larger size. Also, an important application of this result is the design of a “dynamic controller” to stabilize the unstable system $\dot{x}(t)=Ax(t)$.

The paper [95] considers a system of the form

$$\dot{x}(t)=Ax(t)+u(t),\ \ x(t)\in\mathbb{R}^n, \tag{5.1}$$

where $A\in\mathbb{R}^{n\times n}$ with $Tr\,A<0$. The problem of stabilizing (5.1) by state feedback $u(t)=K(t)x(t)$, where $K(t)$ is constrained to be skew-symmetric, is studied.

Ii is shown that the linear system (5.1) can be stabilized by $K(t)$ of the form $K(t)=k(t)S$, where S is a skew-symmetric constant matrix ($S^T=-S$), and $k:\mathbb{R}\to\mathbb{R}$ is a suitable scalar “gain function” (possible a constant) which is sufficiently large. It is derived several stabilization results for the linear system (5.1) by rotation. The concept of “stabilization by rotation” used in this paper encompasses the well-known concept of “vibration stabilization” introduced by

Meerkov [79, 80] and is a deterministic version of "stabilization by noise" for stochastic systems as introduced by L. Arnold and coworkers [96].

In the [97] the non-fragile observer-based control problem for continuous-time linear systems is investigated. Linear matrix inequality approach is used to construct a linear full-order non-fragile observer-based control, which guarantees the exponential stabilization of a closed-loop system.

In the paper [98] an efficient approach to pole placement for the problem of centralized control over a large scale power system with state feedback is proposed. This approach is based on a specific homothetic transformation of the original system representation. The representation contains explicit elements that can be changed by feedback so that to provide a preassigned location of poles of the closed-loop system.

In papers [99, 100] are concerned with pole assignment problem for so-called descriptor system of the form

$$E\{x(t)\} = Ax(t) + Bu(t), \tag{5.2}$$

where $\{x(t)\} \stackrel{\text{def}}{=} dx(t)/dt$ for continuous-time, and $\{x(t)\} \stackrel{\text{def}}{=} x(t+1)$ for discrete-time case of the system (5.2); $x(t)\epsilon\mathbb{R}^n, u(t)\epsilon\mathbb{R}^m, A\epsilon\mathbb{R}^{n\times n}, B\epsilon\mathbb{R}^{n\times m}$ and $E\epsilon\mathbb{R}^{n\times n}$ are constant matrices. The paper [99] presents a method of solving pole assignment problem for the system (5.2) where E is identity matrix with *rank* $E = r$ using derivative state feedback $u(t) = K\{x(t)\}$. The method is based on the decomposition of the system (5.2) with the help of semiorthogonal matrix zero divisors. The paper [100] renders a solution of the following finite eigenvalue assignment problem.

Given the system (5.2,) were matrix E *is singular with rank* $E = r$, *find a matrix* K *such that the system (5.2) closed by state feedback control* $u(t) = K\{x(t)\}$, *i.e. the system* $E\{x(t)\} = (A + BK)x(t)$ *would be asymptotically stable; moreover the set of* r *eigenvalues (poles) of the pair* $(E, A + BK)$ *of matrices* E *and* $A + BK$ *necessarily would have the preassigned form, i.e.*

$$eig(E, A + BK) = \{\lambda_j : det(\lambda_j E - A - BK) = 0,\ j = 1, \dots, r\}.$$

The approach to solving this problem is based on decomposition of the matrix $E = E_L E_R^T$, where $E_L\epsilon\mathbb{R}^{n\times r}, E_R\epsilon\mathbb{R}^{n\times r}$ with

$$rank\, E_L = rank\, E_R = rank\, E = r,$$

and some assertions from matrix theory.

In [101] a novel two-step procedure to design static output feedback controllers is presented. In the first step an optimal stable feedback controller is obtained by means of a linear matrix inequality (LMI). In the second step, a transformation of the LMI variables is used to derive a suitable LMI formulation for the static output feedback controller.

In [102] a simple proportional feedback technique for stabilizing uncertain steady states of dynamical systems is suggested. The method involves either one or two- step algorithm of stabilization. It makes use of either natural or artificially created stable fixed points in order to find the hidden coordinates of the unstable steady state. Two simple mathematical examples are presented and four different physical examples are investigated. Specifically, the mechanical pendulum, the autonomous Duffing damped oscillator, the self-exited van der Pol oscillator and the chaotic Lorenz system with either unknown external forces or unknown control parameters are analytically and numerically considered.

In [103] a stabilization method for *linear time-delay systems* which extends the classical pole assignment method for ordinary differential equations (ODEs) is described. It is shown that the classical pole assignment method for ODEs can be adapted to time-delay systems where the closed-loop system is infinite-dimensional and the number of degrees of freedom of the controller is finite. Unlike methods based on finite spectrum assignment the method proposed does not render the closed-loop system finite-dimensional, but consists of controlling the right-most eigenvalues. The method is explained by means of the stabilization of a linear finite-dimensional system in the presence of an input delay, and its applicability to more general stabilization problems is illustrated. As an illustration of the method proposed the stabilization of the system

$$\dot{x}(t) = Ax(t) + Bu(t-\tau), \tag{5.3}$$

is studied. Here $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}$ is the input, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times 1}$, and $\tau > 0$ represents an input delay. As a control law a linear static state feedback $u(t) = Kx(t), K\in \mathbb{R}^{1\times n}$, is used.

Now we will point out some papers which appear after publications of the original Leonov's and Moreau's & Aeyel's works [30, 71–76], devoted to solving the Brockett's nonstationary stabilization problem.

The paper [104] deals with nonstationary stabilization of a linear single-input single-output time-invariant continuous-time system of the form (1.1) by means of periodic piecewise constant output feedback (4.1) with

$$K(t) = \begin{cases} k_1, \text{if } 0 \le t < T/2, \\ k_2, \text{if } T/2 \le t < T, \end{cases} \quad K(t+T) = K(t),$$

where k_1 and k_2 are constant. By applying averaging theory a sufficient condition is obtained in terms of a bilinear matrix inequality.

In [105] the Brockett stabilization problem is solved for a wide class of systems. Necessary and sufficient conditions for the existence of several classes of stabilizing matrices $K(t)$ are derived. Also a general method of construction of a family of matrices $K(t)$ ensuring stabilization of system (1.1) is described. In this case the stabilization matrix $K(t)$ is arbitrary (not necessary periodic), and the method described is applicable to more general systems than (1.1) when matrices A, B and C are variable: $A = A(t), B = B(t), C = C(t)$.

In the papers [106, 107] the Brocket problem for systems with delay is studied. In [106] necessary and sufficient conditions of asymptotic stabilization of trivial solution of system of nonlinear differential equations of the form

$$\dot{x}(t) = A\big(t, x(t-\tau)\big) + B(t)K(t)C(t), \quad x(t)\in\mathbb{R}^n,$$

are presented. Here $B(t)$ and $C(t)$are given matrices, $A(t,x)$ is a given vector-function, $K(t)$ is unknown stabilization matrix, $\tau > 0$ is a delay.

In [107] the Brocket problem is posed for linear system of the form (1.1) with feedback delay $u(t) = Ky(t-\tau), \tau > 0$. In the paper, the act-and-wait control concept is investigated as a possible technique to reduce the number of poles of systems considered with feedback delay. In this case, the Brockett problem is rephrased for the act-and-wait control system.

6. Conclusions

This paper attempts to survey the state of knowledge concerning the problems of stabilization and pole assignment (placement) for linear continuous time-invariant controllable systems by feedbacks. Different approaches to solving these problems and corresponding main results are presented. The survey includes analytical methods and encompasses both single-input single-output and multi-input multi-output systems. From the studies cited here it is seen that the problems of static output feedback stabilization and pole assignment are still generically open. The existing necessary and/or sufficient conditions are not efficiently testable except for some cases. According to [25] the decision methods are computationally inefficient. Main results of the works [30, 71–74] concerning to solution of the Brockett's stabilization problem are presented. Low- and high-frequency stabilization theorems are formulated. Effective necessary and sufficient analytical conditions for stabilization of two- and three-dimensional systems in terms of system parameters are presented. These conditions show that an introduction a nonstationary feedback control in linear systems extends the possibilities of stationary stabilization. The results presented here can be used in the feedback control of linear systems, and also for stability analysis of nonlinear control systems.

Примечания:

1. Шумафов М.М. Стабилизация неустойчивых состояний равновесия динамических систем. Часть 1 // Вестник Адыгейского государственного университета. Сер. Естественно-математические и технические науки. 2015. Вып. 4 (171). С. 13–21. URL: http://vestnik.adygnet.ru
2. Kailath T. Linear Systems. Englewood Cliffs, H. J.: Prentice-Hall, 1980. 682 pp.
3. Yakubovich V.A., Leonov G.A., Gelig A.Kh. Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities. Singapore: World Scientific, 2004. 350 pp.
4. Леонов Г.А., Шумафов М.М. Методы стабилизации линейных управляемых систем. СПб.: Изд-во СПбГУ, 2005. 420 с.
5. Kalman R.E., Ho Y.C., Narendra K. Controllability of linear dynamical systems // Contr. to Diff. Equations. 1963. Vol. 1. P. 189–213.
6. Леонов Г.А., Шумафов М.М. Элементарное доказательство теоремы о стабилизируемости линейных управляемых систем // Вестник Санкт-Петербургского университета. Сер. Математика, механика и астрономия. 2003. Вып. 3 (17). С. 56–68.
7. Леонов Г.А., Шумафов М.М. Алгоритм пошаговой стабилизации линейного объекта управления // Известия вузов. Северо-Кавказский регион. Естественные науки. 2005. № 2. С. 14–19.
8. Leonov G.A., Shumafov M.M. Stabilization of Linear Systems. Cambridge Scientific Publishers. Cambridge, 2012. 408 pp.
9. Helmke U., Anderson B. Hermitian pencils and output feedback stabilization of scalar systems // Int. J. Control. 1992. Vol. 56. P. 857–876.
10. Algebraic tests for output stabilizability / F. Pérez, C. Abdallah, P. Dorato, D. Docampo // In Proc. 32nd IEEE Conf. on Decision and Control. San Antonio, 1993. P. 2385–2386.
11. Brockett R., Byrnes C. Multivariable Nyquist criteria, root loci, and pole placement: a geometric viewpoint // IEEE Trans. Aut. Control. 1981. AC-26. P. 271–284.
12. Kučera V., Souza C. A necessary and sufficient condition for output feedback stabilizability // Automatica. 1995. Vol. 31. P. 1357–1359.
13. Лурье А.И. Некоторые нелинейные задачи теории автоматического регулирования. М.: Гостехиздат, 1951. 432 с.
14. Лурье А.И. Прямой метод Ляпунова и его применение в теории автоматического регулирования // Труды II Всесоюз. совещания по теории автоматического регулирования. Т. 1. М.: Изд-во АН СССР, 1955. 604 с.
15. Летов А.И. Устойчивость нелинейных регулируемых систем. М.: Физматгиз, 1962. 484 с.
16. Лефшец С. Устойчивость нелинейных систем автоматического управления. М.: Мир, 1967. 183 с.
17. Айзерман М.А., Гантмахер Ф.Р. Абсолютная устойчивость регулируемых систем. М.: Изд-во АН СССР, 1963.139 с.
18. Гантмахер Ф.Р., Якубович В.А. Абсолютная устойчивость нелинейных регулируемых систем // Труды II Всесоюзного съезда по теор. и прикл. механике. М.: Наука, 1965. С. 30–63.
19. Trofino-Neto A., Kučera V. Stabilization via static output feedback // IEEE Trans. Aut. Control. 1993. Vol. AC-38. P. 764–765.

References:

1. Shumafov M.M. Stabilization of unstable steady states of dynamical systems. Part 1 // The Bulletin of Adyghe State University. Ser. Natural-Mathematical and Technical Sciences. 2015. Iss. 4 (171). P. 13–21. URL: http://vestnik.adygnet.ru
2. Kailath T. Linear Systems. Englewood Cliffs, H. J.: Prentice-Hall, 1980. 682 pp.
3. Yakubovich V.A., Leonov G.A., Gelig A.Kh. Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities. Singapore: World Scientific, 2004. 350 pp.
4. Leonov G.A., Shumafov M.M. Method of stabilization of linear controlled systems. SPb.: SPbSU Publishing House, 2005. 420 pp.
5. Kalman R.E., Ho Y.C., Narendra K. Controllability of linear dynamical systems // Contr. to Diff. Equations. 1963. Vol. 1. P. 189–213.
6. Leonov G.A., Shumafov M.M. Elementary proof of the theorem on stabilizability of linear controllable systems // The Bulletin of the Saint-Petersburg University. Ser. Mathematics, Mechanics and Astronomy. 2003. Iss. 3 (17). P. 56–68.
7. Leonov G.A., Shumafov M.M. Algorithm of step-by-step stabilization of linear object of control // News of Higher Schools. North Caucasian Region. Natural Sciences. 2005. No. 2. P. 14–19.
8. Leonov G.A., Shumafov M.M. Stabilization of Linear Systems. Cambridge Scientific Publishers. Cambridge, 2012. 408 pp.
9. Helmke U., Anderson B. Hermitian pencils and output feedback stabilization of scalar systems // Int. J. Control. 1992. Vol. 56. P. 857–876.
10. Algebraic tests for output stabilizability / F. Pérez, C. Abdallah, P. Dorato, D. Docampo // In Proc. 32nd IEEE Conf. on Decision and Control. San Antonio, 1993. P. 2385–2386.
11. Brockett R., Byrnes C. Multivariable Nyquist criteria, root loci, and pole placement: a geometric viewpoint // IEEE Trans. Aut. Control. 1981. AC-26. P. 271–284.
12. Kučera V., Souza C. A necessary and sufficient condition for output feedback stabilizability // Automatica. 1995. Vol. 31. P. 1357–1359.
13. Lurie A.I. Some nonlinear problems of the theory of automatic control. M.: Gostekhizdat, 1951. 432 pp.
14. Lurie A.I. Lyapunov's straight line method and its application in the automatic control theory / Proc. of the 2nd Soviet Union meeting on the automatic control theory. Vol. 1. M.: Publishing House of the USSR AS,1955. 604 pp.
15. Letov A.I. Stability of nonlinear controlled systems. M.: Fizmatgiz, 1962. 484 pp.
16. Lefshetz S. Stability of nonlinear systems of automatic control. M.: Mir, 1967. 183 pp.
17. Ayzerman M.A., Gantmakher F.R. Absolute stability of controlled systems. M .: Publishing House of the USSR AS, 1963. 139 pp.
18. Gantmakher F.R., Yakubovich V.A. Absolute stability of nonlinear controlled systems // Proceedings of the 2nd Soviet Union Congress on theoretical and applied mechanics. M.: Nauka, 1965. P. 30–63.
19. Trofino-Neto A., Kučera V. Stabilization via static output feedback // IEEE Trans. Aut. Control. 1993. Vol. AC-38. P. 764–765.

20. Cao Y.Y., Sun Y.-X., Mao W.-J. A new necessary and sufficient condition for static output feedback stabilizability and comments on “Stabilization via static Output feedback” // IEEE Trans. Aut. Control. 1998. Vol. AC-43 (8). P. 1110–1112.
21. Iwasaki T., Skelton R. Parametrization of all stabilizing controllers via quadratic Lyapunov functions // J. Optim. Theory Applic. 1995. Vol. 77. P. 291–307.
22. El-Chaoui L., Gahinet P. Rank minimization under LMI constraints: a framework for output feedback problems // Proc. Eur. Control Conf. Groningen, 1993. P. 1176–1179.
23. Davison E. An automatic way of finding optional control systems for large multi-variable plants // Proc. IFAC Tokyo Symp. Control. 1965. P. 357–373.
24. Anderson B., Bose N., Jury E. Output feedback stabilization and related problems – solution via decision methods // IEEE Trans. Aut. Control. 1975. Vol. AC-20. P. 53–65.
25. Static output feedback – A Survey / V.L. Syrmos, C.T. Abdallah, P. Dorato, K. Grigoriadis // Automatika. 1997. Vol. 33, No. 2. P. 125–137.
26. Tarski A.A Decision Method for Elementary Algebra and Geometry. 2nd ed. Berkeley: University of California Press, 1951. 68 pp.
27. Seidenberg A. A new decision method for elementary algebra // Ann. Math. 1954. Vol. 64. P. 365–374.
28. Anderson B.D.O., Clements J.C. Algebraic characterization of fixed modes in decentralized control // Automatica. 1981. Vol. 17. P. 703–712.
29. Byrnes C.I., Anderson B.D.O. Output feedback and generic stabilizability // SIAM J. Contr. Optim. 1984. No. 11. P. 362–380.
30. Moreau L., Aeyels D. Periodic output feedback stabilization of single-input single-output continuous-time systems with odd relative degree // Systems & Control Letters. 2004. Vol. 51, No. 5. P. 395–406.
31. Nett C., Bernsfein D., Haddad W. Minimal complexity control law synthesis. Pt. 1: Problem formulation and reduction to optimal static output feedback // Proc. American Contr. Conf. Pittsburg, 1989. P. 2056–2064.
32. Martensson B. The order of any stabilizing regulator is sufficient a priori information for adaptive stabilization // Syst. Control Lett. 1985. Vol. 6. P. 87–91.
33. Квакернаак Х., Сиван Р. Линейные оптимальные системы управления. М.: Мир, 1977. 650 с.
34. Linnemann A. A class of single-input single-output systems stabilizable by reduced-order controllers // Systems Control Lett. No. 11. 1988. P. 27–32.
35. Smith M.C. On minimal order stabilization of single loop plants // Systems Control Lett. 1986. No. 7. P. 39–40.
36. Поляк Б.Т., Щербаков П.С. Трудные задачи линейной теории управления. Некоторые подходы к решению // Автоматика и телемеханика. 2005. № 5. С. 7–46.
37. Зубов В.И. Теория оптимального управления. Л.: Судостроение, 1966. 352 с.
38. Wonham W.M. On pole assignment in multi-input controllable linear systems // IEEE Trans. Aut. Control. 1967. Vol. AC-12 (6). P. 660–675.
39. Калман Р., Фалб П., Арбиб М. Очерки по математической теории систем. М.: Мир, 1971. 400 с.
40. Rissanen J. Control system synthesis by analogue computer based on the “generalized linear feedback control” concept // Proc. Symp. Analog. Computation Applied to the Study Chemical Processes. Intern. seminar. Brussels, 1960. P. 1–13.
41. Rosenbrock H.H., Hayton G.E. The general problems of pole assignment // Int. J. Contr. 1978. Vol. 27, No. 6. P. 837–852.
42. Popov V.M. Hyperstability and optimality of automatic

20. Cao Y.Y., Sun Y.-X., Mao W.-J. A new necessary and sufficient condition for static output feedback stabilizability and comments on “Stabilization via static Output feedback” // IEEE Trans. Aut. Control. 1998. Vol. AC-43 (8). P. 1110–1112.
21. Iwasaki T., Skelton R. Parametrization of all stabilizing controllers via quadratic Lyapunov functions // J. Optim. Theory Applic. 1995. Vol. 77. P. 291–307.
22. El-Chaoui L., Gahinet P. Rank minimization under LMI constraints: a framework for output feedback problems // Proc. Eur. Control Conf. Groningen, 1993. P. 1176–1179.
23. Davison E. An automatic way of finding optional control systems for large multi-variable plants // Proc. IFAC Tokyo Symp. Control. 1965. P. 357–373.
24. Anderson B., Bose N., Jury E. Output feedback stabilization and related problems – solution via decision methods // IEEE Trans. Aut. Control. 1975. Vol. AC-20. P. 53–65.
25. Static output feedback – A Survey / V.L. Syrmos, C.T. Abdallah, P. Dorato, K. Grigoriadis // Automatika. 1997. Vol. 33, No. 2. P. 125–137.
26. Tarski A.A Decision Method for Elementary Algebra and Geometry. 2nd ed. Berkeley: University of California Press, 1951. 68 pp.
27. Seidenberg A. A new decision method for elementary algebra // Ann. Math. 1954. Vol. 64. P. 365–374.
28. Anderson B.D.O., Clements J.C. Algebraic characterization of fixed modes in decentralized control // Automatica. 1981. Vol. 17. P. 703–712.
29. Byrnes C.I., Anderson B.D.O. Output feedback and generic stabilizability // SIAM J. Contr. Optim. 1984. No. 11. P. 362–380.
30. Moreau L., Aeyels D. Periodic output feedback stabilization of single-input single-output continuous-time systems with odd relative degree // Systems & Control Letters. 2004. Vol. 51, No. 5. P. 395–406.
31. Nett C., Bernsfein D., Haddad W. Minimal complexity control law synthesis. Pt. 1: Problem formulation and reduction to optimal static output feedback // Proc. American Contr. Conf. Pittsburg, 1989. P. 2056–2064.
32. Martensson B. The order of any stabilizing regulator is sufficient a priori information for adaptive stabilization // Syst. Control Lett. 1985. Vol. 6. P. 87–91.
33. Kvakernaak Kh., Sivan R. Linear optimal control systems. M.: Mir, 1977. 650 pp.
34. Linnemann A. A class of single-input single-output systems stabilizable by reduced-order controllers // Systems Control Lett. No. 11. 1988. P. 27–32.
35. Smith M.C. On minimal order stabilization of single loop plants // Systems Control Lett. 1986. No. 7. P. 39–40.
36. Polyak B.T., Shcherbakov P.S. Difficult problems of the linear theory of control. Some approaches to the solution // Automatics and Telemechanics. 2005. No. 5. P. 7–46.
37. Zubov V.I. Theory of optimal control. L.: Sudostroenie, 1966. 352 pp.
38. Wonham W.M. On pole assignment in multi-input controllable linear systems // IEEE Trans. Aut. Control. 1967. Vol. AC-12 (6). P. 660–675.
39. Kalman R., Falb P., Arbib M. Essays on mathematical theory of systems. M.: Mir, 1971. 400 pp.
40. Rissanen J. Control system synthesis by analogue computer based on the “generalized linear feedback control” concept // Proc. Symp. Analog. Computation Applied to the Study Chemical Processes. Intern. seminar. Brussels, 1960. P. 1–13.
41. Rosenbrock H.H., Hayton G.E. The general problems of pole assignment // Int. J. Contr. 1978. Vol. 27, No. 6. P. 837–852.
42. Popov V.M. Hyperstability and optimality of automatic

systems with several control functions // Rev. Roum. Sci. Tech. Ser. Electrotech. Energ. 1964. No. 9. P. 629–690.

43. Попов В.М. Гиперустойчивость автоматических систем. М.: Наука, 1970. 453 с.
44. Langenhop C.E. On the stabilization of linear systems // Proc. Amer. Math. Soc. 1964. Vol. 15, № 5. P. 735–742.
45. Chen C.T. A note on pole assignment // IEEE Trans. Aut. Contr. 1968. Vol. AC-12. P. 597–598.
46. Davison E.J. On pole assignment in multivariable linear systems // IEEE Trans. Aut. Contr. 1968. Vol. AC-13. P. 747–748.
47. Heymann M. Comments "On pole assignment in multi-input controllable linear systems" // IEEE Trans. Aut. Contr. 1968. Vol. AC-13. P. 748–749.
48. Simon J.D., Mitter S.K. A theory of modal control // Inform. Contr., 1968. Vol. 13. P. 316–353.
49. Brunovsky P. A classification of linear controllable systems // Kybernetika. 1970. Vol. 6, No. 3. P. 173–187.
50. Аксенов Г.С. К задаче стабилизации линейного объекта управления // Вестник Ленинградского гос. ун-та. 1977. № 7. С. 5–8.
51. Гелиг А.Х., Леонов Г.А., Якубович В.А. Устойчивость нелинейных систем с неединственным состоянием равновесия. М.: Наука, 1978. 400 с.
52. Смирнов Е.Я. Стабилизация программных движений. СПб.: Изд-во СПбГУ, 1977. 306 с.
53. Leonov G.A., Shumafov M.M. Stabilization of linear systems // Proc. 4th European Conf. Structural Control (4ECSC). SPb., 2008. September 8–12. Vol. 2. P. 461–494.
54. Leonov G.A., Shumafov M.M. Stabilization of controllable linear systems // Nonlinear Dynamics and Systems Theory. 2010. Vol. 10, No. 3. P. 235–268.
55. Davison E.J. On pole assignment in multivariable linear systems // IEEE Trans. Aut. Contr. 1971. Vol. AC-16. P. 98–99.
56. Brash F.M., Pearson J.B. Pole placement using dynamic compensator // IEEE Trans. Aut. Contr. 1970. Vol. AC-15 (1). P. 34–43.
57. Davison E.J., Chatterjee R. A note on pole assignment in linear systems with incomplete state feedback // IEEE Trans. Aut. Contr. 1971. Vol. AC-16. P. 98–99.
58. Sridhar B., Lindorff D.P. Pole placement with constant gain output feedback // Int. J. Contr. 1973. Vol. 18. P. 993–1003.
59. Jameson A. Design of a single-input system for specified roots using output feedback // IEEE Trans. Aut. Contr. 1970. Vol. AC-15 (3). P. 345–348.
60. Davison E.J., Wang S.H. On pole assignment in linear multivariable systems using output feedback // IEEE Trans. Aut. Contr. 1975. Vol. AC-20. P. 516–518.
61. Kimura H. On pole assignment by gain output feedback // IEEE Trans. Aut. Contr. 1975. Vol. AC-20. P. 509–516.
62. Kimura H. On pole assignment by output feedback // Int. J. Contol. 1978. Vol. 28. P. 11–22.
63. Shumacher J. Compensator synthesis using (C, A, B) – pairs // IEEE Trans. Aut. Contr. 1980. Vol. AC-25. P. 1133–1138.
64. Rosenthal J. On dynamic feedback compensation and compactification of systems // SIAM J. Contr. Optim. 1994. Vol. 32, No. 1. P. 279–296.
65. Berstein I. On the Lusternik-Šnirel'mann category of real Grassmannians // Proc. Cambr. Phil. Soc. 1976. Vol. 79. P. 129–239.
66. Wang X. Pole placement by static output feedback // Math. Systems, Estimations and Control. 1992. Vol. 2, No. 2. P. 205–218.

systems with several control functions // Rev. Roum. Sci. Tech. Ser. Electrotech. Energ. 1964. No. 9. P. 629–690.

43. Popov V.M. Hyperstability of automatic systems. M.: Nauka, 1970. 453 pp.
44. Langenhop C.E. On the stabilization of linear systems // Proc. Amer. Math. Soc. 1964. Vol. 15, № 5. P. 735–742.
45. Chen C.T. A note on pole assignment // IEEE Trans. Aut. Contr. 1968. Vol. AC-12. P. 597–598.
46. Davison E.J. On pole assignment in multivariable linear systems // IEEE Trans. Aut. Contr. 1968. Vol. AC-13. P. 747–748.
47. Heymann M. Comments "On pole assignment in multi-input controllable linear systems" // IEEE Trans. Aut. Contr. 1968. Vol. AC-13. P. 748–749.
48. Simon J.D., Mitter S.K. A theory of modal control // Inform. Contr., 1968. Vol. 13. P. 316–353.
49. Brunovsky P. A classification of linear controllable systems // Kybernetika. 1970. Vol. 6, No. 3. P. 173–187.
50. Aksenov G.S. On the problem of stabilization of the linear object of control // Bulletin of Leningrad State University. 1977. No. 7. P. 5–8.
51. Gelig A.H., Leonov G.A., Yakubovich V.A. Stability of nonlinear systems with a nonunique equilibrium state. M.: Nauka, 1978. 400 pp.
52. Smirnov E.Ya. The stabilization of program motions. SPb.: SPbSU Publishing House, 1977. 306 pp.
53. Leonov G.A., Shumafov M.M. Stabilization of linear systems // Proc. 4th European Conf. Structural Control (4ECSC). SPb., 2008. September 8–12. Vol. 2. P. 461–494.
54. Leonov G.A., Shumafov M.M. Stabilization of controllable linear systems // Nonlinear Dynamics and Systems Theory. 2010. Vol. 10, No. 3. P. 235–268.
55. Davison E.J. On pole assignment in multivariable linear systems // IEEE Trans. Aut. Contr. 1971. Vol. AC-16. P. 98–99.
56. Brash F.M., Pearson J.B. Pole placement using dynamic compensator // IEEE Trans. Aut. Contr. 1970. Vol. AC-15 (1). P. 34–43.
57. Davison E.J., Chatterjee R. A note on pole assignment in linear systems with incomplete state feedback // IEEE Trans. Aut. Contr. 1971. Vol. AC-16. P. 98–99.
58. Sridhar B., Lindorff D.P. Pole placement with constant gain output feedback // Int. J. Contr. 1973. Vol. 18. P. 993–1003.
59. Jameson A. Design of a single-input system for specified roots using output feedback // IEEE Trans. Aut. Contr. 1970. Vol. AC-15 (3). P. 345–348.
60. Davison E.J., Wang S.H. On pole assignment in linear multivariable systems using output feedback // IEEE Trans. Aut. Contr. 1975. Vol. AC-20. P. 516–518.
61. Kimura H. On pole assignment by gain output feedback // IEEE Trans. Aut. Contr. 1975. Vol. AC-20. P. 509–516.
62. Kimura H. On pole assignment by output feedback // Int. J. Contol. 1978. Vol. 28. P. 11–22.
63. Shumacher J. Compensator synthesis using (C, A, B) – pairs // IEEE Trans. Aut. Contr. 1980. Vol. AC-25. P. 1133–1138.
64. Rosenthal J. On dynamic feedback compensation and compactification of systems // SIAM J. Contr. Optim. 1994. Vol. 32, No. 1. P. 279–296.
65. Berstein I. On the Lusternik-Šnirel'mann category of real Grassmannians // Proc. Cambr. Phil. Soc. 1976. Vol. 79. P. 129–239.
66. Wang X. Pole placement by static output feedback // Math. Systems, Estimations and Control. 1992. Vol. 2, No. 2. P. 205–218.

67. Willems J.C., Hesselink W.H. Generic properties of the pole placement problem // Proc. of the 7th IFAC Congress. 1978. P. 1725–1729.
68. Rosenthal J., Willems J.C. Open problems in the area of pole placement // Open Problems in Mathematical Systems and Control Theory / ed. by V.D. Blondel, E.D. Sontag, M. Vidyasagar, J.C. Willems. London: Springer, 1999. P. 181–191.
69. Andry A.N., Shpiro E.Y., Chung J.C. Eigenstructure Assignment for Linear Systems // IEEE Trans. Aerospace & Electronic Systems. 1983. Vol. Aes-19, No. 5. P. 711–728.
70. Brockett R. A stabilization problem // Open Problems in Mathematical Systems and Control Theory / ed. by V.D. Blondel, E.D. Sontag, M. Vidyasagar, J.C. Willems. London: Springer, 1999. P. 75–78.
71. Леонов Г.А., Шумафов М.М. Проблемы стабилизации линейных управляемых систем. СПб.: Изд-во СПбГУ, 2002. 307 с.
72. Леонов Г.А. Проблема Брокетта в теории устойчивых линейных дифференциальных уравнений // Алгебра и анализ. 2001. Т. 13, вып. 4. С. 134–155.
73. Леонов Г.А. Стабилизационная проблема Брокетта // Автоматика и телемеханика. 2001. № 5. С. 190–193.
74. Leonov G.A. The Brockett Problem in the Theory of Nonstationary Stabilization of Linear Differential Equations // Amer. Math. Soc. Transl. 2002. Vol. 205, No. 2. P. 163–173.
75. Moreau L., Aeyels D. Stabilization by means of periodic output feedback // Proc. of conference of decision and control (CDC). Phoenix, 1999. December 7–10. P. 108–109.
76. Moreau L., Aeyels D. A note on stabilization by periodic output feedback for third-order systems // Proc. of the 14th International Symposium of Mathematical Theory of Networks and Systems (MTNS). Perpignan, 2000. 19–23 June. 2 pp.
77. Андреев Ю.Н. Управление конечномерными линейными объектами. М.: Наука, 1976. 424 с.
78. Арнольд В.И. Обыкновенные дифференциальные уравнения М.: Наука, 1971. 239 с.
79. Меерков С.М. Вибрационное регулирование // Автоматика и телемеханика. 1973. № 2. Р. 34–43.
80. Meerkov S.M. Vibrational control theory // J. Franklin Inst. 1977. Vol. 3, No. 2. P. 117–128.
81. Stephenson A. A new type of dynamical stability // Mem. Proc. Manchr. Cit. Phil. Soc. 1908. Vol. 52. P. 1907–1908.
82. Капица П.Л. Динамическая устойчивость маятника при колеблющейся точке подвеса // Журн. экспер. и теор. 1951. Т. 21, вып. 5. С. 588–598.
83. Боголюбов Н.Н. Теория возмущений в нелинейной механике // Сборник института строительной механики АН УССР. 1950. Т. 14. С. 9–34.
84. Byrnos C.I. Pole placement by output feedback. In three decades of mathematical system theory // Lecture notes in control and information science. Vol. 135 / ed. by H. Hijmeijer, J.M. Schumacher., N. Y.: Springer, 1989. P. 31–78.
85. Bernstein D.S. Some Open Problems in Matrix Theory Arising in Linear Systems and Control // Linear Algebra and its Applications. 1992. Vol. 162–164. P. 409–432.
86. Open Problems in Mathematical Systems and Control Theory / ed. by V.D. Blondel, E.D. Sontag, M. Vidyasagar, J.C. Willems. London: Springer, 1999. 300 pp.
87. Blondel V., Gevers M., Lindquist A. Survey on the state of the systems and control // European J. Contol. 1995. Vol. 1. P. 5–23.
88. Кузовков Н.Т. Модальное управление и наблюдающие устройства. М., 1976. 184 с.

67. Willems J.C., Hesselink W.H. Generic properties of the pole placement problem // Proc. of the 7th IFAC Congress. 1978. P. 1725–1729.
68. Rosenthal J., Willems J.C. Open problems in the area of pole placement // Open Problems in Mathematical Systems and Control Theory / ed. by V.D. Blondel, E.D. Sontag, M. Vidyasagar, J.C. Willems. London: Springer, 1999. P. 181–191.
69. Andry A.N., Shpiro E.Y., Chung J.C. Eigenstructure Assignment for Linear Systems // IEEE Trans. Aerospace & Electronic Systems. 1983. Vol. Aes-19, No. 5. P. 711–728.
70. Brockett R. A stabilization problem // Open Problems in Mathematical Systems and Control Theory / ed. by V.D. Blondel, E.D. Sontag, M. Vidyasagar, J.C. Willems. London: Springer, 1999. P. 75–78.
71. Leonov G.A., Shumafov M.M. Problems of stabilization of linear controlled systems. SPb.: SPbSU Publishing House, 2002. 307 pp.
72. Leonov G.A. Brockett's problem in the theory of stability of the linear differential equations // Algebra and Analysis. 2001. Vol. 13, Iss. 4. P. 134–155.
73. Leonov G.A. Brockett's stabilization problem // Automatics and Telemechanics. 2001. No. 5. P. 190–193.
74. Leonov G.A. The Brockett Problem in the Theory of Nonstationary Stabilization of Linear Differential Equations // Amer. Math. Soc. Transl. 2002. Vol. 205, No. 2. P. 163–173.
75. Moreau L., Aeyels D. Stabilization by means of periodic output feedback // Proc. of conference of decision and control (CDC). Phoenix, 1999. December 7–10. P. 108–109.
76. Moreau L., Aeyels D. A note on stabilization by periodic output feedback for third-order systems // Proc. of the 14th International Symposium of Mathematical Theory of Networks and Systems (MTNS). Perpignan, 2000. 19–23 June. 2 pp.
77. Andreev Yu.N. Control of finite-dimensional linear objects. M.: Nauka, 1976. 424 pp.
78. Arnold V.I. Ordinary differential equations. M.: Nauka, 1971. 239 pp.
79. Meerkov S.M. Vibration regulation // Automatics and Telemechanics. 1973. No. 2. P. 34–43.
80. Meerkov S.M. Vibrational control theory // J. Franklin Inst. 1977. Vol. 3, No. 2. P. 117–128.
81. Stephenson A. A new type of dynamical stability // Mem. Proc. Manchr. Cit. Phil. Soc. 1908. Vol. 52. P. 1907–1908.
82. Kapitsa P.L. Dynamic stability .of the pendulum at the oscillating point of suspension // Journal. of Exper. and Theory. 1951. Vol. 21, No. 5. P. 588–598.
83. Bogolyubov N.N. Perturbation theory in nonlinear mechanics // Proceedings of the Institute of Building Mechanics of the UkrSSR AS. 1950. Vol. 14. P. 9–34
84. Byrnos C.I. Pole placement by output feedback. In three decades of mathematical system theory // Lecture notes in control and information science. Vol. 135 / ed. by H. Hijmeijer, J.M. Schumacher., N. Y.: Springer, 1989. P. 31–78.
85. Bernstein D.S. Some Open Problems in Matrix Theory Arising in Linear Systems and Control // Linear Algebra and its Applications. 1992. Vol. 162–164. P. 409–432.
86. Open Problems in Mathematical Systems and Control Theory / ed. by V.D. Blondel, E.D. Sontag, M. Vidyasagar, J.C. Willems. London: Springer, 1999. 300 pp.
87. Blondel V., Gevers M., Lindquist A. Survey on the state of the systems and control // European J. Contol. 1995. Vol. 1. P. 5–23.
88. Kuzovkov N.T. Modal control and watching devices. M., 1976. 184 pp.

89. Зайцев В.А. Модальное управление линейным дифференциальным уравнением с неполной обратной связью // Дифф. уравнения. 2003. Т. 39, № 1. С. 133–135.
90. Зайцев В.А. Управление спектром в линейных системах с неполной обратной связью // Дифф. уравнения. 2009. Т. 45, № 9. С. 1320–1328.
91. Зайцев В.А. Управление спектром в билинейных системах // Дифф. уравнения. 2010. Т. 46, № 7. С. 1061–1064.
92. Зайцев В.А. Необходимые и достаточные условия в задаче управления спектром // Дифф. уравнения. 2010. Т. 46, № 12. С. 1789–1793.
93. Yang K., Orsi R. Generalized pole placement via based on projections // Automatica. 2006. Vol. 42. P. 2143–2150.
94. Wu J.-W., Lian K.-Y., Lancaster P. A stabilization criterion for matrices // Linear Algebra and its Applications. 2007. Vol. 422. P. 22–28.
95. Crauel H., Damm T., Ilchmann A. Stabilization of linear systems by rotation // J. Diff. Equations. 2007. Vol. 234. P. 412–438.
96. Arnold L., Crauel H., Wihstutz V. Stabilization of linear systems by noise // SIAM J. Cont. Optim. 1983. Vol. 21. P. 451–461.
97. Lien Ch.-H., Cheng W.-Ch., Tsai Ch.-H., Yu K.-W. Non-fragile observer-based controls of linear system via LMI approach // Chaos, Solitons and Fractals. 2007. Vol. 32. P. 1530–1537.
98. Мисриханов М.Ш., Рябченко В.Н. Размещение полюсов при управлении большой энергетической системой // Автоматика и телемеханика. 2011. № 10. С. 129–153.
99. Размещение полюсов при управлении MIMO-системой с обратной связью по производным состояния / Н.Е. Зубов, Е.А. Микрин, М.Ш. Мисриханов, В.Н. Рябченко // Вестник МГТУ им. Н.Э. Баумана. Сер. Приборостроение. 2015. № 4. С. 3–12.
100. Управление конечными собственными значениями дескрипторной системы / Н.Е. Зубов, Е.А. Микрин, М.Ш. Мисриханов, В.Н. Рябченко // Докл. акад. наук. 2015. Т. 460, № 4. С. 381–384.
101. Recent advances in Static Output Feedback Controller Design with Applications to Vibration Control of Large Structures / F. Palacios-Quiñonero, J. Rubio'-Massegu', J.M. Rossel, H.R. Kapimi // Modeling, Identification and Control. 2014. Vol. 35, No. 3. P. 169–190.
102. Tamaševičiūte E., Tamaševičius A. Stabilizing uncertain steady states of some dynamical systems by means of proportional feedback // Nonlinear Analysis: Modeling and Control. 2013. Vol. 18, No. 1. P. 86–98.
103. Continuous pole placement for delay equations / W. Michiels, K. Engelborghs, P. Vansevenant, D. Roose // Automatica. 2002. Vol. 38. P. 747–761.
104. Allwright J.C., Astolfi A., Wong H.P. A note on asymptotic stabilization of linear systems by periodic, piecewise constant, output feedback // Automatica. 2005. Vol. 41. P. 339–344.
105. Бойков И.В. К стабилизационной проблеме Брокетта // Автоматика и телемеханика. 2005. № 5. С. 76–82.
106. Бойков И.В. Проблема Брокетта для систем нелинейных дифференциальных уравнений с запаздыванием // Известия вузов. Поволжский регион. Физико-математические науки. 2011. № 4 (20). С. 3–13.
107. Insperger T., Stepan G. Brockett problem for systems with feedback delay // Proc. of the 17th World Congress of the Intern. Fed. of Aut. Contr. (IFAC). Seoul, 2008. July 6–11. P. 11491–11496.

89. Zaytsev V.A. Modal control of the linear differential equation with incomplete feedback // Diff. Equations. 2003. Vol. 39, No. 1. P. 133–135.
90. Zaytsev V.A. Spectrum control in linear systems with incomplete feedback // Diff. Equations. 2009. Vol. 45, No. 9. P. 1320–1328.
91. Zaytsev V.A. Spectrum control in bilinear systems // Diff. Equations. 2010. Vol. 46, No. 7. P. 1061–1064.
92. Zaytsev V.A. Necessary and sufficient conditions in the spectrum control problem // Diff. Equations. 2010. Vol. 46, No. 12. P. 1789–1793.
93. Yang K., Orsi R. Generalized pole placement via based on projections // Automatica. 2006. Vol. 42. P. 2143–2150.
94. Wu J.-W., Lian K.-Y., Lancaster P. A stabilization criterion for matrices // Linear Algebra and its Applications. 2007. Vol. 422. P. 22–28.
95. Crauel H., Damm T., Ilchmann A. Stabilization of linear systems by rotation // J. Diff. Equations. 2007. Vol. 234. P. 412–438.
96. Arnold L., Crauel H., Wihstutz V. Stabilization of linear systems by noise // SIAM J. Cont. Optim. 1983. Vol. 21. P. 451–461.
97. Lien Ch.-H., Cheng W.-Ch., Tsai Ch.-H., Yu K.-W. Non-fragile observer-based controls of linear system via LMI approach // Chaos, Solitons and Fractals. 2007. Vol. 32. P. 1530–1537.
98. Misrikhanov M.Sh., Ryabchenko V.N. The placement of the poles in the control of large energy systems // Automatics and Telemechanics. No. 10. P. 129–153.
99. The placement of the poles in the control of the MIMO-feedback system on derivative status / N.E. Zubov, E.A. Mikrin, M.Sh. Misrikhanov, V.N. Ryabchenko // Bulletin of MSTU named after N.E. Bauman. Ser. Instrument Engineering. 2015. No. 4. P. 3–12.
100. Control of finite eigen values of the descriptor system / N.E. Zubov, E.A. Mikrin, M.Sh. Misrikhanov, V.N. Ryabchenko // Reports of the Academy of Sciences. 2015. Vol. 460, No. 4. P. 381–384.
101. Recent advances in Static Output Feedback Controller Design with Applications to Vibration Control of Large Structures / F. Palacios-Quiñonero, J. Rubio'-Massegu', J.M. Rossel, H.R. Kapimi // Modeling, Identification and Control. 2014. Vol. 35, No. 3. P. 169–190.
102. Tamaševičiūte E., Tamaševičius A. Stabilizing uncertain steady states of some dynamical systems by means of proportional feedback // Nonlinear Analysis: Modeling and Control. 2013. Vol. 18, No. 1. P. 86–98.
103. Continuous pole placement for delay equations / W. Michiels, K. Engelborghs, P. Vansevenant, D. Roose // Automatica. 2002. Vol. 38. P. 747–761.
104. Allwright J.C., Astolfi A., Wong H.P. A note on asymptotic stabilization of linear systems by periodic, piecewise constant, output feedback // Automatica. 2005. Vol. 41. P. 339–344.
105. Boykov I.V. On the stabilization problem of Brockett // Automatics and Telemechanics. 2005. No. 5. P. 76–82.
106. Boykov I.V. Brockett's problem for systems of nonlinear differential equations with delay // Proceedings of the Universities. The Volga region. Physics and Mathematics. 2011. No. 4 (20). P. 3–13.
107. Insperger T., Stepan G. Brockett problem for systems with feedback delay // Proc. of the 17th World Congress of the Intern. Fed. of Aut. Contr. (IFAC). Seoul, 2008. July 6–11. P. 11491–11496.