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Stabilization of unstable steady states of dynamical systems.* Part 3. Stabilization by time-delayed feedback control. – A survey**

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Abstract. *The paper is continuation of two previous papers (Part 1 and Part 2) of the present work. In the Part 1 a short survey on the feedback control stabilization of unstable steady states of dynamical systems is presented, and stabilization problem statements are formulated. In the Part 2 a survey on stationary and nonstationary stabilization and pole assignment for linear control systems is presented. Here, in the Part 3 of the work, a survey on delayed feedback stabilization is given. The results concerning to stabilization of unstable steady states of two-dimensional dynamical systems are presented. Necessary and/or sufficient conditions in the terms of system parameters are given. These conditions show that the introduction a delay feedback control law in the system in general extends the possibilities of stabilization by feedback without delay. The results can be used for stability analysis of nonlinear chaotic systems in the neighborhood of unstable steady states.*

Keywords: *controllable system, unstable steady state, asymptotic stability, stabilization, delay feedback control.*

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Стабилизация неустойчивых состояний равновесия динамических систем.*** Часть 3. Стабилизация обратной связью с запаздыванием. – Обзор****

(Рецензирована)

Аннотация. *Настоящая статья является продолжением двух предыдущих статей: части 1 и части 2 данной работы. В первой части был сделан краткий обзор работ по стабилизации неустойчивых состояний равновесия управляемых динамических систем, были сформулированы постановки задач. Во второй части был сделан обзор результатов работ по стационарной стабилизации линейных управляемых систем, а также работ по проблеме управления спектром матрицы (назначение полюсов). Здесь, в третьей части, дан обзор работ по стабилизации неустойчивых состояний равновесия обратной связью с запаздыванием. Приведены результаты работ по стабилизации неустойчивых состояний равновесия двумерных динамических систем. Даны необходимые и/или достаточные условия стабилизируемости в терминах параметров систем. Эти условия показывают, что введение в систему обратной связи с запаздыванием расширяет в целом возможности стабилизации при отсутствии запаздывания в обратной связи. Результаты могут быть использованы при исследовании устойчивости состояний равновесия нелинейных хаотических динамических систем.*

Ключевые слова: *управляемая система, неустойчивое состояние равновесия, асимптотическая устойчивость, стабилизация, управление обратной связью с запаздыванием.*

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** *This work is continuation of papers [1, 2].*

*** *Работа представляет собой расширенный текст пленарного доклада на Первой Международной научной конференции «Осенние математические чтения в Адыгее», посвященной памяти профессора К.С. Мамия, 8–10 октября 2015 г. Адыгейский государственный университет, Майкоп, Республика Адыгея.*

**** *Данная работа является продолжением статей [1, 2].*

1. Introduction

This paper is continuation of papers [1] and [2]. In [1] a short survey on the feedback control stabilization of unstable steady states (unstable equilibria) of dynamical systems is given, stabilization and pole assignment problem statements (Problems 1, 2 and 3) for linear controllable systems are formulated. In [2] these problems are discussed, and a survey on stationary and nonstationary stabilization (Brockett's problem), and pole assignment for linear control systems is presented.

In the present paper a survey of works on stabilization of unstable steady states of dynamical systems by time-delayed feedback control is given. We consider the stabilization problem by means of delay feedback control which is formulated in [1] as Problem 4.

In recent years a growing interest in the stability of time-delay systems is appeared in the control literature. The interest for this is motivated by applications in control systems, physics, chemistry and biology. A review on results and methods in stability analysis of time-delay systems can be found in, e.g., [3–8].

In the applications one often deals with the control of systems by means of an introduction time-delayed feedback in order to achieve noninvasive stabilization. A new powerful motivation for employment of time-delayed feedback in the control of systems is the problems of controlling chaos and its stabilization. Starting with the original works of Ott, Grebogi, Yorke [9] and K. Pyragas [10] this problem is intensively studied by numerous researchers for last more than twenty years (see, for instance, surveys [11–14]). In this works stabilization of chaotic system is achieved by stabilization of unstable periodic orbits (UPOs) embedded in a strange attractor of the system. For this purpose a simple and powerful effective method was proposed by Pyragas in [10].

This method, called *delayed feedback control (DFC)*, consist in constructing a control law in such a way that the control input is proportional to the difference of the output (or state) of the system delayed by some time in the past. The DFC algorithm is successfully applied to many real problems in physical, chemical, and biological systems, in particular, in electronic chaotic oscillators, mechanical pendulums, lasers, cardiac systems etc.

It turned out that the DFC method which was originally invented for stabilization of UPOs is also suitable to stabilize unstable steady states (USSs) of dynamical systems. According to Pyragas the problem of stabilizing USSs by DFC scheme "is, maybe, more important for various applications than the problem of stabilizing UPOs".

It should be noted that although the effects of DFC (and its different extensions) techniques are studied in numerous works (see, for instance, bibliography in [13, 14]), much less is known for stabilization of USSs. We will point out the works on stabilization of USSs by DFC along with the literature review (see References).

In the papers [15–17] Pyragas studied stabilization of USSs dynamical systems. In [15] theoretical and experimental results of stabilizing an USS in a special so-called Mackey-Class system, described by the first order delay differential equation are presented. (This eqution is a model for regeneration of the blood cells in patient with leukemia). These results show the efficiency of the feedback techniques to the stabilize the USS of the Mackey-Class system. In [16] a simple adaptive controller based on low-pass filter to stabilize of USSs of dynamical systems is considered. In these case it does not require knowledge of the location of the fixed point in the phase spase. In [17] an adaptive dynamic state feedback controller for stabilizing and tracking unknown USSs of dynamical systems is proposed.

In the Ushio's paper [18] the problem of stabilization of unstable states of nonlinear discrete-time systems by means of Pyragas' DFC is studied. In this work Ushio established that the DFC can stabilize only a certain class of unstable steady states, and thus it was shown that the possibilities of Pyragas DFC law are limited. Analogous fact was established for continuous-time systems by Kokame et al. [19].

In [20] Hövel and Schöll computed the stabilization domain of an unstable focus in the plane parameterized by feedback gain and time delay, and also presented analytical solution of a characteristic equation using the special Lambert function. In [21] Yanchuk et al. performed asymptotic analysis of DFC for a large delay time. The parameter ranges for stabilization of the unstable focus point is established.

In [22] Dahms et al. show that extended DFC can be used to stabilize an unstable state of focus type. It is shown that this modification DFC method is able to control a larger region of unstable state in comparison with the original DFC scheme.

In [23, 24] Ahlborn and Parlitz proposed another modification of DFC, a namely, a multiple delay feedback control to stabilize USSs. In contrast to extended DFC delay times are not integer multiplies of a time delay in Pyragas's DFC, but may enter independently in control terms. It is shown that the multiple DFC is more effective for stabilization of USSs compared to DFC and extended DFC. This approach is demonstrated by stabilizing USSs with the help of numerical simulations of different chaotic dynamical systems (as Chua's circuit, Rössler, Lorenz systems).

In [25] Gjurchinovski and Urumov proposed a variable DFC to improve the performance of DFC in stabilizing USSs. In this approach the delay time is not constant during the control process, but it is variable and modulated in time in a suitable way. The method proposed is illustrated by a numerical simulation of the Rössler and Lorenz systems. The authors perform a comparative analysis between their method and the original Pyragas' DFC method by using USS a focus type as an example. It is shown that if the additional parameter introduced is positive, then the region of control is enlarged in comparison with that which yields DFC.

In [26] Guzenko et al. apply the adaptive modification of DFC to the stabilization of USS and UPO embedded in a chaotic attractor.

In the work [8] Hijmeijer et al. studied the stabilization problem for two-dimensional dynamical systems by means of Pyragas' DFC. For solving this problem the authors proposed a method which is based on the eigenvalues optimization approach. This method in conjunction with numerical simulation made possible to "guess" in most cases the analytical expressions for the boundary of the stabilization regions.

Different from [8] approach to solving the USSs stabilization problem by means of DFC is proposed in the papers [27–32]. This approach is based on the D-decomposition method first suggested by Yu.I. Neimark [33]. The idea of this method consists in decomposition of space of the system parameters into regions in which the characteristic equation associated with a USS has the same number of roots with positive real part, and consequent choice of stability regions. The advantage of this approach is that the investigation of the problem, firstly, is purely analytical, and secondary, the proofs of the theorems are simpler. In this case necessary and/or sufficient conditions of stabilizability of USSs by DFC of two-dimensional dynamical systems are obtained.

2. Mathematical Formulation of DFC

We will begin with the statement of the general stabilization problem.

2.1. General Stabilization Problem by DFC

Consider a continuous-time nonlinear control system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ y(t) = g(x(t)), \end{cases} \tag{2.1}$$

where $x(t) \in \mathrm{R}^n$ is a state vector, $u(t) \in \mathrm{R}^m$ is an input (control) vector, $y(t) \in \mathrm{R}^l$ is an output vector, the function f and g are differentiable. The point over the symbol x denotes a differentiation in t.

Suppose that the origin $x = 0$ is a steady state, (i.e. $f(0,0) = 0,\ g(0) = 0$) which is unstable and hyperbolic (i.e. the matrix of the linearized about $x = 0$ system has no eigenvalues located on imaginary axis).

The problem consists in finding a control

$$u(t) = h(y(t), y(t-\tau)), \quad h(0,0) = 0, \tag{2.2}$$

such that the steady state of closed-loop system (2.1), (2.2)

$$\dot{x}(t) = f(x(t), h(g(x(t)), g(x(t-\tau)))) \tag{2.3}$$

would be asymptotically stable.

For solving this problem as in, e.g., [8, 20, 21] linear stability analysis is performed.

2.2 Stabilization problem for linear system by DFC

The linearization of system (2.1) about point $(x,u) = (0,0)$ yields

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{2.4}$$

where $A = D_x f(0,0)$, $B = D_u f(0,0)$, $C = D_x g(0)$. Here D_x and D_u are differentiation operators with respect to x and u, respectively; $A \in \mathrm{R}^{n\times n}$, $B \in \mathrm{R}^{n\times m}$, $C \in \mathrm{R}^{l\times n}$ are the Jacobian matrices of functions f and g, evaluated at the points $(x,u) = (0,0)$ and $x = 0$, respectively.

Without loss of generality we assume that rank $A = m$, rank $C = l$. Consider two manners of introducing the time delay feedback (2.2):

$$u(t) = Ky(t-\tau) \quad \text{(classical DFC)} \tag{2.5}$$

and

$$u(t) = -K[y(t) - y(t-\tau)] \quad \text{(Pyragas' DFC)}, \tag{2.6}$$

where $K \in \mathrm{R}^{m\times l}$ and τ is a positive parameter.

The closed-loop systems (2.4), (2.5) and (2.4), (2.6) have the forms

$$\dot{x}(t) = Ax(t) + BKCx(t-\tau), \tag{2.7}$$

$$\dot{x}(t) = Ax(t) + BKC[x(t-\tau) - x(t)], \tag{2.8}$$

respectively. (The systems (2.7) and (2.8) are the linearized systems of the closed-loop system (2.3), where h is defined by (2.5) and (2.6), respectively.)

Now we formulate linear stabilization problem by DFC (Problem 4. Delayed Feedback Control, see [1]):

Given a system (2.4), find necessary and/or sufficient conditions under which there exist an real $(m\times l)$-matrix K and a number $\tau > 0$ such that the origin of the system (2.7)/(2.8) would be asymptotically stable.

Since systems (2.7) and (2.8) are linear, every solution $x = \xi(t)$ of the system (2.7) ((2.8)) is asymptotically stable (then we say that the system is asymptotically stable) if and only if the trivial solution $x(t) \equiv 0$ is asymptotically stable. Therefore, in what follows we will sometimes say the system (instead of origin $x = 0$) is asymptotically stable.

Reformulation of Stabilization Problem by DFC

It is known [3-5] that the origin of the system (2.7) and (2.8) is *asymptotically stable* if and only if all the roots of their characteristic equations

$$det\left(\lambda I - A - BKCe^{-\tau\lambda}\right) = 0 \quad (\lambda \in \mathrm{C}), \tag{2.9}$$

$$det\left[\lambda I - (A - BKC) - BKCe^{-\tau\lambda}\right] = 0, \tag{2.10}$$

respectively, have *negative real parts*, i.e. are located in the left half-plane of the complex plane. Here I is $(n\times n)$-identity matrix.

Hence, the stabilization problem for linear system (2.4) can be reformulated in the following way:

Find necessary and/or sufficient conditions under which there exists a matrix $K \in \mathrm{R}$ and a number $\tau > 0$ such that the real part of all roots of characteristic equation (2.9)/(2.10) would be negative.

By virtue of the stability theorem on first approximation the origin $x = 0$ of nonlinear system (2.3) is asymptotically stable if all roots of characteristic equation of the linearized system of the system (2.3) around $x = 0$ located in the left half-plane of complex plane ([4, 5]). Hence, the origin $x = 0$ of the system (2.1), (2.5)/(2.1), (2.6) is asymptotically stable if all roots of the characteristic equation (2.9)/(2.10) have negative real parts.

Therefore, we focus on stabilization of unstable steady state $x = 0$ of *linear* system (2.4) by DFC (2.5)/(2.6).

2.3. Two-dimensional case $(n = 2)$

Assume that $m = l = 1$ $(u(t) \in \mathrm{R},\ y(t) \in \mathrm{R})$ and the linearized system (2.4) is controllable. Then the system (2.4) can be reduced to the canonical form [34]

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -a_1 x_1(t) - a_2 x_2(t) - u, \\ y = c_1 x_1 + c_2 x_2, \end{cases} \tag{2.11}$$

where a_1, a_2, c_1, c_2 are some real constants. Characteristic equations of the system closed by feedback controls (2.5) and (2.6) (where K is a number) take the forms

$$\lambda^2 + a_2\lambda + a_1 + ke^{-\tau\lambda}(c_2\lambda + c_1) = 0 \quad (k := K), \tag{2.12}$$

$$\lambda^2 + (a_2 - kc_2)\lambda + (a_1 - kc_1) + ke^{-\tau\lambda}(c_2\lambda + c_1) = 0 \quad (k := K), \tag{2.13}$$

respectively.

The problem consists in finding conditions on parameters a_1, a_2, c_1 *and* c_2 *under which there exist numbers* $k \neq 0$ *and* $\tau > 0$ *such that all roots of the equation (2.12)/(2.13) would have negative real parts.*

Thus, the study of stabilization problem is reduced to the problem of an analysis of the disposition of roots of characteristic equations in the left half-plane of complex plane. Note the left parts of characteristic equations (2.9), (2.10) and (2.12), (2.13) are quasipolynomials, and therefore analytical functions. A large number of publications were devoted to the problem of studying the zeros of quasipolynomials, and, generally, entire functions. The classical works on this topic are due to L.S. Pontryagin [35] and N.G. Chebotarev, N.N. Meiman [36].

3. Main Results

The theory of stabilization of systems by DFC (and its extensions) is rather difficult since the closed-loop system is a delay differential equation. As for linear delay differential equations (2.7) and (2.8), so here the linear stability analysis is also complicated due to the infinite number of the roots of their transcendental characteristic equations (2.9) and (2.10). Nevertheless some analytical results are obtained in works [8, 11, 12, 15–32, 37–39].

In this section we present main results obtained by analytical methods with the help of DFC (2.5) and (2.6). Here we will chiefly focus on Pyragas' DFC (2.6) as an effective and powerful method of the stabilization of chaotic systems.

3.1. Odd Number Limitation (ONL)

As is noted above in Section 1 the possibilities of DFC (2.6) for stabilization of dynamical systems are limited. Ushio [18] was the first who discovered this fact. When Ushio studied the stabilization of unstable steady states (USSs) of discrete-time systems by Pyragas' DFC, he found that the DFC is subject to a substantial limitation, which is now referred to as the odd number limitation (ONL). Later the limitation of Pyragas' DFC for stabilization of USSs of continuous-time systems was derived by Kokame et al. [19]. After that, analogous limitation of the DFC was found for stabilization of unstable periodic orbits (UPOs) of continuous-time systems by Just et al. [37] and Nakajima [38, 39] for non-autonomous systems, and by Just et al. [37] and Hootan, Amann [40] for autonomous system. Similar results were obtained for some other cases by Tian et al. [41, 42], Hino et al. [43], Morgül [44].

Below we formulate the ONL results of Ushio [18] and Kokame et al. [19].

Consider a discrete analog of the system (2.1) in which the output is a state $(y(t) = x(t))$, i.e. a discrete-time system

$$x(k+1) = f(x(k), u(k)), \quad (y(k) = x(k)), \quad (k \in \mathrm{N}), \tag{3.1}$$

for which the DFC has a form (discrete analog to (2.6))

$$u(k) = -K[x(k) - x(k-1)], \; K \in \mathrm{R}^{m \times n}. \tag{3.2}$$

Let $x = 0$ be an unstable steady state of (3.1) without control: $f(0,0) = 0$. The linearized system around the point $x = 0$ is

$$x(k+1) = Ax(k) + Bu(k), \tag{3.3}$$

where $A = D_x f(0,0)$, $B = D_u f(0,0)$.

Theorem 3.1 (ONL for stabilizing USSs of discrete-time systems [18])**.** *If the Jacobi's matrix A of the system (3.1), evaluated at the USS point $x = 0$, has either an eigenvalue which is equal to the unity or an odd number of real eigenvalues that are greater than unity, then the linearized system of the closed-loop system around point $x = 0$, denoted by (A, B), cannot be stabilized by the DFC (3.12) with any choice of the constant feedback gain matrix K.*

Since

$$det(I - A) = \prod_{j=1}^{n} (1 - \lambda_j(A)), \tag{3.4}$$

it is easy to see that the real matrix A has an odd number of real eigenvalues which are greater than unity if and only if the inequality holds

$$det(I - A) < 0$$

provided that all the eigenvalues $\lambda_j(A)$ of the matrix A are different from the unity: $\lambda_j \neq 1$ $(j = 1, \ldots, n)$. From (3.4) it is also obvious that the matrix A has an eigenvalues being equal to the unity if and only if the equality $det(I - A) = 0$ holds. Therefore the assertion of Theorem 3.1 can be reformulated as follows:

$$(A, B) \textit{ is stabilizable via Pyragas' DFC } (3.2) \Rightarrow det(I - A) > 0. \tag{3.5}$$

The proof of statement (3.5) is easy. Indeed, the linearized system of the closed-loop system (3.1), (3.2) is

$$x(k+1) = Ax(k) + BK[x(k-1) - x(k)]. \tag{3.6}$$

The characteristic polynomial $d(\lambda)$, associated with (3.6), is

$$d(\lambda) = det[\lambda^2 I - \lambda(A - BK) - BK]. \tag{3.7}$$

Since the system (3.6) should be asymptotically stable, all the roots of polynomial (3.7) are located inside the unit circle centered at the origin. The latter implies $d(1) > 0$ irrespectively of K by the stability criterion e.g., [3-5]) for linear discrete systems. Really, otherwise if $d(1) \leq 0$, then from continuity of $d(\lambda)$ and the relation $\lim_{\lambda \to +\infty} d(\lambda) = +\infty$ it follows that there exist a real number $\lambda_0 \geq 1$ ($\lambda_0 = 1$ in the case $d(1) = 0$) such that $d(\lambda_0) = 0$, i.e. the polynomial (3.7) have a root lying outside of the unit disc (or on the unit circle). Therefore, $d(1) > 0$. On the other hand $d(1) = det(I - A)$ by (3.7). From here it follows (3.5).

Now, we formulate odd number limitation for continuous-time system (2.1).

Theorem 3.2 (ONL for stabilizing USSs of continuous-time systems [19])**.** *If the Jacobi's matrix A of the system (2.1), evaluated at the USS point $x = 0$, has either an eigenvalue which is* *equal to zero or an odd number of real eigenvalues that are greater than zero, then the linearized system of the closed-loop system around point* $x = 0$, *denoted by* (A, B), *cannot be stabilized by the DFC (2.6), where* $y(t) = x(t)$, *with any choices of the constant feedback gain matrix* K *and positive number* τ.

Since

$$det(-A) = (-1)^n \prod_{j=1}^{n} \lambda_j(A), \tag{3.8}$$

under the condition that all the eigenvalues of the matrix A are different from zero ($\lambda_j \neq 0$), the real matrix A has an odd number of real eigenvalues if and only if the inequality holds : $det(-A) < 0$. Also, from (3.8) it follows that matrix A has an eigenvalue being equal to zero if and only if $det(-A) = 0$. Therefore, the Theorem 3.2 can be reformulated in the following way:

$$(A, B) \text{ is stabilizable via Pyragas' DFC } (2.6) \Rightarrow det(-A) > 0. \tag{3.9}$$

The assertion (3.9) can be also easily proved as follows. The characteristic quasipolynomial, associated with the closed-loop system (2.8) (where C is the identity matrix, since $y(t) = x(t)$ in (2.6)) is

$$d(\lambda) = \left[\lambda I - (A - BK) - BKe^{-\tau\lambda}\right]. \tag{3.10}$$

Since the closed-loop system (2.8) ($C = I$) should be asymptotically stable, all the roots of quasipolynimial (3.10) lie in the open left half-plane of complex plane. By analogy with (3.7), as above, the latter implies $d(0) > 0$ for (3.10). On the other hand from (3.10) we can see that $d(0) = det(-A)$. Hence, (3.9) is proved.

Remark 3.1. As is seen from proof of the Theorem 3.2 the assertion of this theorem remains to be true for output feedback (2.6) (the case $C \neq I$ and $l < n$). The same is true for Theorem 3.1: its assertion is also true for output feedback $u(k) = -K[y(k) - y(k-1)]$, where $y(k) = Cx(k)$, $C \in \mathrm{R}^{l \times n}$. By (3.9) ((3.5)) the inequality $det(-A) > 0$ $(det(I - A) > 0)$ is a *necessary condition* for the stabilizability of continuous-time linear system (2.4) (discrete-time linear system (3.3)) via Pyragas' DFC (2.6) ((3.2)). Thus, the odd number limitation describes a necessary condition for the stabilizability via Pyragas' DFC.

Remark 3.2. There is also odd number limitation for stabilizing unstable periodic orbits (UPOs) of the systems via Pyragas' DFC law. In this case instead of the eigenvalues of the matrix A in Theorem 3.1 and Theorem 3.2, the eigenvalues (Floquet multipliers) of the state transition matrix (monodromy matrix) $\phi(T,0)$ (T is a period of UPO) are considered for discrete-time [43] and nonautonomous continuous-time systems [37–39]. For autonomous continuous-time systems ONL for stabilizing UPOs is proved by Hooton and Amann [40].

3.2. Overcoming the odd number limitation

An importment problem in the study of stabilization of USSs (and also UPOs) is how to overcome the odd number limitation (ONL). After publication of Nakajima's paper [38] there were suggested various modified DFC schemes by many researches in order to overcome the ONL. One of these schemes was a modification of the DFC, which was proposed by Ushio [45] for discrete-time systems. In order to overcome the ONL a prediction-based control law for stabilization of USS of the discrete-time system (3.1) was suggested. This control law is

$$u(k) = K[x(k) - f(x(k), 0)]. \tag{3.11}$$

It is obvious that the control signal (3.11) tends to zero if the system trajectory converges to a USS of the system (3.1). It was proved the following

Theorem 3.3 (Ushio [45])**.** *If the linearized system (3.3) is stabilizable, then the steady state* $x = 0$ *of nonlinear system (3.1) is stabilizable by (3.11) if and only if*

$$det(I - A) \neq 0 .$$

Other approach for overcoming the ONL is based on converting the linearized system (3.3), (3.2) to an augmented system without delay by introducing the matrices

$$\overline{A} = \begin{pmatrix} 0 & I \\ 0 & A \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}, \quad \overline{C} = \begin{pmatrix} -I & I \end{pmatrix}$$

with $z(k) = (x(k-1), \ x(k))^T$.

Then the DFC method can be regarded as a special output feedback control law $u(k) = Ky(k)$ for the augmented system

$$\begin{cases} z(k+1) = \overline{A}z(k) + \overline{B}u(k), \\ y(k) = \overline{C}z(k). \end{cases}$$

Therefore, the ONL can be considered as the limitation in the problem of stabilization by output feedback control $u = Ky$. Under the assumption that the pair $(\overline{A}, \overline{C})$ is observable and $(\overline{A}, \overline{B})$ is stabilizable, Konishi and Kokame [46] derived an observer-based dynamic delayed-feedback control law for stabilization of USSs of discrete-time systems.

As for continuous-time systems overcoming the ONL is rather difficult because the dimension of the phase space of closed-loop system with DFC is infinite.

To overcome the ONL Nakajima [47] extended the prediction-based DFC, proposed originally for discrete-time systems, to continuous-time systems.

In [6] Pyragas proposed another method to overcome the ONL. The main idea of this approach is to introduce into the feedback loop an additional unstable degree of freedom that changes the total number of unstable real positive modes to an even number. This idea was applied to the construction of a simple adaptive controller for stabilizing unknown steady states of dynamical systems. From the Pyragas' results it follows that any unstable steady state with real positive eigenvalues cannot be stabilized with a stable controller. More precisely, to stabilize the USS $x = 0$ one is introduced an adaptive feedback

$$u(t) = kQ\dot{w}(t), \tag{3.12}$$

where $w \in \mathrm{R}^s$ is a dynamical variable of the controller, that satisfies the equation

$$\dot{w}(t) = Pw(t) + Ry(t), \quad y(t) = Cx(t). \tag{3.13}$$

Here in (3.12) and (3.13) $P \in \mathrm{R}^{s \times s}$, $Q \in \mathrm{R}^{m \times s}$ and $R \in \mathrm{R}^{s \times l}$ are matrix parameters, and k is a scalar parameter, which are to be determined. In this case the closed-loop system (2.1), (3.12), (3.13), i.e. the system

$$\begin{cases} \dot{x} = f(x, kQ(Pw + RCx)), \\ \dot{w} = Pw + RCx \end{cases} \tag{3.14}$$

is considered in the phase space R^{n+s} variables $x, \ w$.

Note that the feedback (3.12) vanishes whenever the steady state $(x = 0, w = 0)$ of the closed-loop system (3.14) is stabilized. The following statement holds.

Theorem 3.4 (Pyragas [6]). *Let P be a nonsingular matrix in (3.13). If the total number of real positive eigenvalues of the Jacobi's matrix $J = D_x f(0,0)$ and the matrix P is odd, then the steady state $(0, 0)$ of the closed-loop system (3.14) cannot be stabilized by any choice of matrices $P, \ Q, \ R$ and control gain k.*

From Theorem 3.4 it follows that if the Jacobi's matrix J at USS has an odd number of real positive eigenvalues, then it can be stabilized only with an *unstable* controller whose matrix P has an odd number of real positive eigenvalues.

3.3. Necessary and Sufficient Conditions

In this section we dwell on the question of under which conditions on the system parameters there exist a time-delayed state feedback such that the unstable steady state is asymptotically stable.

Consider the linear system (2.4), having a scalar input, i.e. $B \in \mathrm{R}^{n\times 1}$ $(m=1)$. Assume that the output $y(t)$ of the system (2.4) is state $x(t)$, i.e. $y(t)=x(t)$ $(C=I)$. Then the DFC (2.6) takes a form

$$u(t) = -K[x(t) - x(t-\tau)], \tag{3.15}$$

where K is a constant gain row-matrix to be determined.

In the discrete-time case Ushio [18] obtained a necessary and sufficient condition for the first-order and second-order controllable systems. For high-order single-input systems necessary and sufficient conditions were obtained in [42].

Theorem 3.5 (Zhu & Tian [42]). *Suppose that (A,B) is controllable. Then there exists a DFC (3.2), where $K \in \mathrm{R}^{1\times n}$ $(m=1)$ such that the closed-loop system (3.3), (3.2), is asymptotically stable if and only if*

$$0 < det(I - A) < 2^{n+1}. \tag{3.16}$$

In the cases $n=1$ and $n=2$ the condition (3.16) is exactly the same as the result obtained by Ushio [18].

If the pair (A, B) is not controllable, without loss of generality, it may assume that (A, B) has the following controllability decomposition form [34]

$$A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}, \tag{3.17}$$

where A is a matrix of dimension $n_1 \times n_1$, A_2 is $(n_2 \times n_2)$-matrix, b_1 is a column-vector of dimension n_1 $(n_1 + n_2 = n)$. For this case the necessary and sufficient conditions are formulated as follows.

Theorem 3.6 (Zhu&Tian). *Assume that (A, B) is uncontrollable. Then the system (3.3), (3.17), denoted by (A, B), is stabilizable by DFC (3.2), where $K \in \mathrm{R}^{1\times n}$, if and only if:*

1) the matrix A_2 is stable, i.e. $\lambda_j(A_2) < 0$ $(j = \overline{1, n_2})$;

2) $0 < det(I - A_1) < 2^{n_1+1}$.

Now we turn to continuous-time systems. Kokame et al. [19] considered the stabilization of USSs for single-input continuous-time systems (2.4), where $m=1$ and $C=I$, by delayed state feedback control (3.15). It was shown that if the odd number limitation is excluded, then the stabilization of USSs by the feedback (3.15) is possible. The exact statement of this result is as follows.

Theorem 3.7 (Kokame et al. [19]). *Suppose that in the nonlinear system (2.1) $u(t)$ is a scalar input $(m=1)$ and $g(x) \equiv x$ $(y(t) \equiv x(t))$. Assume that the linearized system of (2.1), denoted by (A, B), about the USS $x=0$ is controllable and the $det(-A) > 0$. Then there exists a pair of $K \in \mathrm{R}^{1\times n}$ and $\tau > 0$ such that the closed-loop system (2.4), (3.15), i.e. the system (2.8), where $B \in \mathrm{R}^{n\times 1}$, $C = I$, is asymptotically stable.*

Combining the assertion of the Theorem 3.7 with (3.9) we have the following statement.

Theorem 3.8 (Kokame et al. [19]). *Assume that the linear system (2.4), where $B \in \mathrm{R}^{n\times 1}(m=1)$, $C = I$, denoted by (A, B), is controllable. Then (A, B) is stabilizable via DFC (3.15) <=> $det(-A) > 0$.*

Since $det(-A)$ is a constant term of the characteristic polynomial $det(\lambda I - A)$ of the matrix A, from Theorem 3.8 if follows

Corollary. *Let* $d(\lambda)=\lambda^n+a_n\lambda^{n-1}+\ldots+a_1$ *be a characteristic polynomial of the matrix* A *of the system (2.4). Then under the assumptions of the Theorem 3.8 for the system (2.4)* $(m=1,\ C=I)$ *to be stabilizable by DFC (3.15) it is necessary and sufficient that* $a_1>0$.

In view of Remark 3.1 we also obtain the following

Proposition 3.1. *Let* $d(\lambda)=\lambda^n+a_n\lambda^{n-1}+\ldots+a_1$ *be a characteristic polynomial of the matrix* A *of system (2.4). Then for the system (2.4) to be stabilizable by DFC (2.6) it is necessary that* $a_1>0$.

3.4. Two-dimensional systems

Here we present stabilization criteria of unstable steady states of two-dimensional continuous-time dynamical systems.

Consider the single-input single-output control system (2.11). We will formulate necessary and sufficient conditions of stabilization of the system (2.11) by DFCs (2.5) and (2.6), where K and $\tau>0$ are scalar parameters.

By Corollary of Theorem 3.8 we have

Proposition 3.2. *For the system (2.11) to be stabilizable by delayed state feedback control (3.15) it is necessary and sufficient that* $a_1>0$.

From Proposition 3.1 it follows

Proposition 3.3. *For the system (2.11) to be stabilizable by delayed output feedback control (2.6) it is necessary that* $a_1>0$.

It what follows we consider three cases:

1) $c_1\neq 0,\ c_2=0$; 2) $c_2\neq 0,\ c_1=0$; 3) $c_1\neq 0,\ c_2\neq 0$.

The following theorems yield necessary and/or sufficient conditions for stabilization of the system (2.11) by feedback (2.5) and (2.6).

Theorem 3.9 [28, 30]. *Assume* $c_1\neq 0,\ c_2=0$ *in (2.11). The system (2.11) is stabilizable by DFC (2.5) if and only if*

a) $a_2>0$ *or b)* $a_1>a_2^2/2,\ a_2\leq 0$.

Theorem 3.10 [27, 29, 31, 32]. *Assume* $c_1\neq 0,\ c_2=0$. *The system (2.11) is stabilizable by DFC (2.6) if and only if* $a_1>0$.

From the Theorem 3.10 it follows:

Corollary 3.1. *Let the origin* $x=0$ *of open system (2.11)* $(u=0;\ c_1\neq 0,\ c_2=0)$ *be an unstable node or focus. Then there exist numbers* K *and* $\tau>0$ *such that the origin is stabilizable by the DFC (2.6).*

Corollary 3.2. *Let the origin* $x=0$ *of open system (2.11)* $(u=0;\ c_1\neq 0, c_2=0)$ *be a saddle point. Then the system (2.11) cannot be stabilized by the DFC (2.6) with any choice of the constant feedback gain* $K\in\mathrm{R}$ *and time-delay* $\tau>0$.

From Theorem 3.9 it follows that the DFC (2.5) can stabilize only the saddles determined by condition $a_2>0$, and unstable focuses, determined by condition b) of Theorem 3.9.

Theorem 3.11 [28, 30]. *Assume that* $c_1=0,\ c_2\neq 0$. *Then the system (2.11) is stabilize by DFC (2.5) if and only if the inequality* $a_1>0$ *holds.*

Theorem 3.12 [27, 29, 31, 32]. *Assume* $c_1=0,\ c_2\neq 0$. *Then for stabilizability of the system (2.11) by the DFC (2.6) it is necessary and sufficient that at least one of the following conditions is satisfied:*

1) $a_1>0,\ a_2>0$;

2) $a_1 > \dfrac{(m\pi a_2)^2}{16}, \quad a_2 \le 0,$

where

$$m = \min_{\alpha\in[0,\,2\pi]}\left(\cos\alpha + \frac{\sin\alpha}{\alpha}\right) \quad (m \approx 1{,}0419).$$

From Theorem 3.12 it follows that the unstable node or focus of system (2.11) $(c_1 = 0, c_2 \ne 0)$ is stabilizable by DFC (2.6) if and only if the condition 2), where $a_2 < 0$, is fulfilled. But stabilization of saddle point is impossible by DFC (2.6) with any choice of parameters $K \in \mathrm{R}$ and $\tau > 0$.

Theorem 3.13 [28, 30]. *Suppose that* $c_1 \ne 0, c_2 \ne 0$ *in (2.11). Then necessary and sufficient condition that the system (2.11) be stabilizable by DFC (2.5) is that at least one of the following conditions is satisfied:*

1) $c_1 c_2 > 0$*;*

2) $c_1 c_2 < 0,\ a_1 > c_1 a_2 / c_2$*;*

3) $c_1 c_2 > 0,\ a_1 > 0,\ \dfrac{c_2}{c_1}\sqrt{a_1\left[a_1 + 2\left(\dfrac{c_1}{c_2}\right)^2\right]} < a_2 < \dfrac{c_2 a_1}{c_1}.$

In the special case $c_1 = -1,\ c_2 = 1$ the stabilizability condition of the system (2.11) by (2.5) takes one of the forms:

a) $a_1 + a_2 > 0$ or b) $a_1 + a_2 \le 0,\ a_1 + a_2 \ne -1,\ (a_1+1)^2 - a_2^2 > 1$.

Theorem 3.14 [27, 29, 31, 32]. *Suppose that* $c_1 \ne 0, c_2 = 1$ *in (2.11). Then sufficient condition for the system (2.11) to be stabilizable by DFC (2.6) is that at least one of the following is satisfied:*

1) $c_1 > 0,\ a_1 > 0$*;*

2) $c_1 < 0,\ a_1 > 0,\ a_2 > c_1$*;*

3) $c_1 < 0, a_1 > c_1 a_2 + (\pi c_1 a_2)^2/4\beta^2, \quad -\infty < a_2 \le \dfrac{2m_1\beta^2}{m\beta - 2c_1},\ 0 < \beta < 2c_1/m,$

where m *is the number from Theorem 3.12, and*

$$m_1 = \min_{\alpha\in[0,\,2\pi]}(\sin\alpha)/\alpha \quad (\approx -0{,}2173).$$

Remark 3.3. In the case $c_1 > 0$ the condition 1) of the Theorem 3.14 is also necessary.

The Theorem 3.14 implies that DFC (2.6) can stabilize unstable nodes or focuses of the system (2.11) in the case $c_1 > 0$. If $c_1 < 0$ their stabilization is possible under condition 2) or 3) of Theorem 3.14. In the case of saddle point the stabilization by DFC (2.6) is impossible with any choice of parameters $K \in \mathrm{R}$ and $\tau > 0$.

Remark 3.4. For three-dimensional systems ($n = 3$) the results are announced in [48]. Here we note that in this case the necessary and sufficient condition that the three-dimensional system be stabilized by delayed state feedback $u(t) = -K[(x(t) - x(t-\tau))]$, where K is (1×3)-real matrix and $\tau > 0$, is that the inequality $a_1 > 0$ holds.

Remark 3.5. Various modifications of the Pyrags' DFC (2.6) are proposed in order to improve its possibility for stabilization of USSs and UPOs. One of such modifications was suggested by Socolar et al. in the paper [49], in wich the feedback is taken in the form of infinite series by using an information from many previous states of the system. This method is known as extented delay feedback control (EDFC). The ONL assertion of Theorem 3.2 remains true also in the case of EDFC [50].

4. Conclusion

In this paper a review on stabilization of unstable steady states(USSs) of dynamical systems by time-delayed feedback control is presented. We mainly consider the original Pyragas' delayed feedback control (DFC), which has been widely used in controlling chaos. Stability analysis including the well-known odd number limitation (ONL) of the DFC is reviewed. Some developments in characterizing the ONL of the DFC and modified DFC schemes, developed in order to overcome the ONL are presented. Necessary and/or sufficient conditions for stabilizability of USSs of two-dimensional systems by delayed state feedback and delayed output feedback controls are given. These conditions also show the possibilities of the DFC methodology for stabilization of linear controllable systems.

The stabilization theorems for two-dimensional linear systems, presented in the section 3.4, well illustrate how the introduction a delay in the system feedback opens up additional possibilities. The conditions given by these theorems generically enlarge the domain of stability of USS in the space of system parameters derived by ordinary static ouput feedback without delay.

Stabilization of USSs as well as UPOs by Pyragas' delayed feedback control and its various modification is still an active area of research. There remain many significant and interesting theoretical as well as technical problems wich require futher studying. Some open questions among others are the following [11]:

1. Necessary and sufficient condition of stabizability via DFC for general multi-input discrete-time systems.
2. Converting the problem of stabilizability of UPOs via DFC to the stabilizability of USSs via DFC for multi-input continuous-time systems.
3. Constructive designing a DFC for a continuous-time systems without requiring information about the target UPO.
4. Robustness of DFC and the design of robust DFC for uncertain chaotic systems.
5. Bifurcation analysis of the closed-loop system with DFC in terms of the delay constant and the control gain.

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