# МАТЕМАТИКА

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## Existence, uniqueness, stability and oscillations of solutions of differential equations with hysteresis nonlinearities. – A survey

*(Peer-reviewed)*

***Abstract.*** *A survey on results for ordinary differential equations with hysteresis nonlinearities is presented. The results concerning the problems of existence and/or uniqueness of solutions for the Cauchy initial value problem for various classes of differential equations with hysteresis operators are provided. Frequency conditions of the absolute stability and existence of oscillatory solutions of equations with nonlinearities of different types of hysteresis are given.*
***Keywords:*** *differential equation, hysteresis nonlinearity, existence and uniqueness, stability, oscillatory solution, frequency condition.*

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## Существование, единственность, устойчивость и колебательность решений дифференциальных уравнений с гистерезисными нелинейностями. – Обзор

*(Рецензирована)*

***Аннотация.*** *Дан обзор результатов работ о свойствах решений обыкновенных дифференциальных уравнений с гистерезисными нелинейностями. Представлены результаты, касающиеся существования и единственности решения начальной задачи Коши для различных классов уравнений с гистерезисными операторами. Получены частотные условия абсолютной устойчивости и существования колебательных решений уравнений с гистерезисными нелинейностями различных типов.*
***Ключевые слова:*** *дифференциальное уравнение, гистерезисная нелинейность, существование и единственность, устойчивость, колебательное решение, частотное условие.*

## 1. Introduction

The theory of differential equations with hysteresis nonlinearities is a new and exciting branch of modern science. It came into being in the early 1960s starting from pioneering works of Russian mathematician V.A. Yakubovich [1, 2].

Hysteresis is a nonlinear phenomenon that occurs in physics, chemistry, biology, engineering, economics. In physics for instance we encounter it in ferromagnetism, ferroelectricity, plasticity, friction, adsorption and desorption. Hysteresis also appears in phase transitions in porous media filtration, smart materials, semiconductors, mechanical damage, and so on.

The term "hysteresis" originates from ancient Greek and means "to lag behind" or "coming behind". It was first introduced into the scientific vocabulary by the Scottish physicist Alfred Ewing [3] in 1885.

In [4], three important properties of hysteresis have been stated as main its characteristics: *lagging*, *memory* and *rate-independence*.

The *lagging* property means that the output lags behind the input. The *memory* property says that the current output depends not only the current input, but also on the history of the input. The *rate-independence* property means that the input-output map does not depend on the frequency of the input, but only on the amplitude of the input.

Mathematical models of hysteresis were first developed by Prandtl [5] and Preisach [6] in the early 20th century. Further research on hysteresis were based on phenomenological approach and mostly focused on modeling and characterization of physical hysteresis. As far as we know it was only in 1963 that hysteresis was given a first functional approach by V.A. Yakubovich [1], regarding hysteresis as a map between functional spaces. In [1] hysteresis was defined as collection of operators. Later the mathematical theory of ordinary differential equations coupled with hysteresis operators was systematically developed in 1970s due to the works of Russian mathematicians Krasnoselskii and his co-workers.

In the period 1970–1980 Krasnoselskii, Pokrovskii and others conducted a systematic analysis of the mathematical properties of hysteresis in terms of operators, acting in functional spaces. They published a number of papers, which formed the basis for the monograph [7] of Krasnoselskii and Pokrovskii.

It should be pointed out that according to Visintin [4] an engineering student R. Bouc [8] was one of the first, "who modelled several phenomena, regarding hysteresis as a map between function spaces".

In the 1980s some western applied mathematicians also began to study mathematical models of hysteresis, especially in connection with applications. For more information about these research we refer the reader to the fundamental monographs of Visintin [4], Brokate and Sprekels [9], Krejčí [10], Mayergoyz [11] and collective volumes [12]. We also mention the surveys [13–16].

## 2. Definition of hysteresis operator

In the literature, there are a multitude of mathematical definitions of hysteresis, and as it is noted in [16], "the terminology for, and precise definition of, hysteresis has varied from area to area and paper to paper". Here me mention the four most commonly used definitions of hysteresis operator.

Although the hysteresis phenomenon has been discovered in magnetism about 150 years ago, the first strict mathematical definition of hysteresis has been provided, as it is pointed out in Introduction, by Yakubovich [1] in 1963. In [1] hysteresis operator is defined as a family of operators that act among the suitable function spaces and possess certain properties (in details, see [1]).

Later, another strict mathematical definition of hysteresis operator was given by Krasnoselskii and his co-workers [17, 18] in 1970. They defined a basic hysteresis operator, called a *hysteron*. For this they used a geometric approach. As a preliminary, they at first define a hysteresis operator called a "play" or "backlash", and after that *generalized play* operator. The complete description of the operator-hysteresis and continuous systems of such hysterons, and results on related topics are presented in the monograph [7] of Krasnoselskii and Pokrvskii (for details we refer to [7]).

In [9] Brokate and Sprekels proposed different approach to define a hysteresis operator. This

approach is based on the notion of *strings*. They use the transitions between functions and strings, that is, between continuous and discrete input informations. At the beginning, the authors introduce the *rate independence functional,* which play the fundamental role in their construction. After that they show that their notion of a hysteresis operator has exactly the desired properties, which are inherent to hysteresis phenomenon.

The fourth definition of a hysteresis operator has been provided by Visintin in his fundamental monograph [4] (see, also [19]), in which hysteresis operator is defined as an operator, having the *memory property and rate-independent* one.

**Definition 1** [9]. *A function* $\varphi: [0,T] \to [0,T]$ *is called a time transformation if* $\varphi(t)$ *is continuous, increasing and satisfies*

$$\varphi(0) = 0, \quad \varphi(T) = T.$$

**Definition 2** [4, 19]. *An operator* $W$ *is called rate independence if*

$$W[u \circ \varphi;\ u_0, w_0] = W[u; u_0, w_0] \circ \varphi$$

*for any admissible time transformation* $\varphi: [0,T] \to [0,T]$, *i.e. for any fixed* $u_0, w_0$,

$$W[\cdot\,; u_0, w_0]: u \to w \Rightarrow W[\cdot\,; u_0, w_0]: u \circ \varphi \to w \circ \varphi.$$

This means that at any instant time $t$, $w(t)$ only depends on $u([0,t])$ and on the order in which values have been attained before $t$. Thus, at any instant time $t$, the value $w(t)$ (output) depends on the *previous* values of $u$ (input) and on the initial state of the system. It can be expressed as follows

$$w(t) = W[u(\tau)|_0^t;\ u_0, w_0](t) \quad \forall t \in [0,T],$$

where $W[\cdot\,; u_0, w_0]$ represents an operator that depends on parameters $u_0, w_0$, and acts among suitable spaces of time-dependent functions $\{u: [0,T] \to \mathbb{R}\}$ and $\{w: [0,T] \to \mathbb{R}\}$. Here $u_0 = u(0)$,

$$W[u(\tau)|_0^t;\ u_0, w_0](0) = w_0, \quad (u_0, w_0) \in H.$$

The following concept of causality is an important property of hysteresis.

**Definition 3** [4, 19]. *An operator* $W$ *is said to be causal if for any* $t \in [0,T]$, *the output* $w(t)$ *is independent of* $u|_{[t,T]}$, *i.e.*

$$u_1|_{[0,t]} = u_2|_{[0,t]} \Rightarrow W[u_1; u_0, w_0](t) = W[u_2; u_0, w_0](t).$$

The operator $W$ is also called a *memory* or *Volterra operator*.

Based on the definition of the rate-independent and causality, one can given the Visintin's definition of hysteresis operator.

**Definition 4** [4, 19]. *An operator* $W$ *is called an hysteresis operator if it is causal and rate-independent.*

Note that for an operator $W$ it is also required to be fulfilled the following *semigroup property*:

$$W[u; u_0, w_0](t_2) = W[u(t_1 + \cdot); u_1, w_1](t_2 - t_1),$$

where $u_1 = u(t_1)$, $w_1 = W[u; u_0, w_0](t_1)$.

The question of how these four definitions are related to each other remains open.

## 3. Differential equations with hysteresis operators

A simple example, which leads to differential equation with hysteresis, is an iron pendulum oscillating in the external magnetic field. The pendulum dynamics is described in the linear approximation by the equation [20]

$$\ddot{x} + k\dot{x} + x = A\sin(\omega t) + \xi(t), \qquad \xi(t) = \varphi[x](t),$$

where the nonlinearity $\varphi[x](t)$ describes the interaction between the external magnetic field and the magnetized pendulum, and $k$, $A$, $\omega$ are some positive numbers. The $\varphi[x](t)$ depends both on the pendulum position and pendulum magnetization. The pendulum's current magnetization de-

pends in turn on the previous history of the pendulum motion. Thus, we have the equation of the differential-hysteresis operator type.

### 3.1. Existence and uniqueness: Cauchy initial value problem

Consider a differential-operator system

$$\begin{cases} \dot{x} = f(t, x, \xi(t)), & (1) \\ \xi(t) = \varphi\left[x(\tau)|_{\tau=t_0}^{t}; t_0, x_0, \xi_0\right](t), & (2) \end{cases}$$

with the initial conditions

$$x(t_0) = x_0, \quad \xi(t_0) = \xi_0, \tag{3}$$

where $x = x(t)$, $\xi = \xi(t)$ are scalar continuous functions on $[t_0, T] \subset \mathbb{R}$, $f(t, x, \xi)$ is a function of their arguments $t \in [t_0, T]$, $x, \xi \in \mathbb{R}$, and $\varphi[\cdot\,; t_0, x_0, \xi_0]$ is some hysteresis operator, acting in each space $C[t_0, t_1]$ $(t_0 < t_1 \leq T)$.

Suppose that

$$\varphi[x_0; t_0, x_0, \xi_0](t) \equiv \xi_0, \qquad \varphi\left[x(\tau)|_{t_0}^{t}; t_0, x_0, \xi_0\right](t_0) = \xi_0.$$

We say the pair $(x(t), \xi(t))$ is a *solution* of the initial value problem (1)–(3) on an interval $[t_0, t_1]$ if:

a) $x \in C^1[t_0, t_1]$, $\xi \in C[t_0, t_1]$, $t_1 \in (t_0, T]$;

b) functions $x(t)$ and $\xi(t)$ satisfy the equation (1) on interval $[t_0, t_1]$;

c) functions $x(t)$ and $\xi(t)$ on $[t_0, t_1]$ are related by equality (2).

The following theorem is analog of the classical Cauchy-Lipschitz-Picard-Lindelöf theorem.

**Theorem 1** [21]. *Suppose the functions $f$ and operator $\varphi$ satisfy the following conditions:*

1) *function $f$ is continuous of $(t, x, \xi)$ in a domain $D = [t_0, T] \times G$, where $G \subset \mathbb{R}^2_{x,\xi}$;*

2) *function $f$ satisfies on the domain $D$ to Lipschitz condition with respect to arguments $x$ and $\xi$:*

$$|f(t, x_1, \xi_1) - f(t, x_2, \xi_2)| \leq L(|x_1 - x_2| + |\xi_1 - \xi_2|)$$

*for some constant $L > 0$ and any two points $(t, x_1, \xi_1)$ and $(t, x_2, \xi_2)$ in $D$;*

3) *operator $\varphi$ is defined on the set $H[t_0, t_1]$ of functions $x(\cdot)$, $x(t_0) = x_0$, satisfying to Lipschitz condition on $[t_0, t_1]$, and maps $H[t_0, t_1]$ into $C[t_0, t_1]$;*

4) *operator $\varphi$ satisfies to Lipschitz condition:*

$$\left|\varphi\left[x_1(\tau)|_{t_0}^{t}; t_0, x_1(t_0), \xi_0\right](t) - \varphi\left[x_2(\tau)|_{t_0}^{t}; t_0, x_2(t_0), \xi_0\right]\right| \leq\leq K \max_{t_0 \leq \tau \leq t} |x_1(\tau) - x_2(\tau)|$$

*for $\forall t \in [t_0, T]$ and for any two functions $x_1(t)$ and $x_2(t)$ from the domain of definition of operator $\varphi$, $K > 0$ is a constant.*

*Then the initial value problem* (1)–(3) *has a unique solution, defined on some interval $[t_0, t_0 + d]$, where $d > 0$ is some sufficiently small number.*

The proof of Theorem 1 is analogous to proof of Cauchy-Lipschitz-Picard-Lindelöf theorem. For this purpose it is used the contraction mapping principle.

The following theorem is analog of the classical Peano existence theorem.

**Theorem 2** [21]**.** *Suppose that:*

1) *function $f(t, x, \xi)$ satisfies the condition* 1) *of Theorem* 1*;*

2) *operator $\varphi$ maps the set $C[t_0, t_1]$ of continuous functions $x(t)$, $x(t_0) = x_0$, on $[t_0, t_1]$ into the set $C[t_0, t_1]$ of continuous functions $\xi(t)$, $\xi(t_0) = \xi_0$ on $[t_0, t_1]$, and satisfies the Lipschitz condition* 4) *of Theorem* 1.

*Then the Cauchy initial problem* (1)–(3) *has at least one solution, defined on interval $[t_0, t_0 + d]$, where $d > 0$ is some sufficiently small number.*

The proof of Theorem 2 is based on transition from differential equation (1) to corresponding integral one and consequent applying of Schauder's existence fixed point principle.

Now we formulate the analog of classical Osgood uniqueness theorem.

**Theorem 3** [22]. *Suppose:*

1) $f(t,x,\xi)$ *is continuous function of* $(t,x,\xi)$ *for* $(t,x,\xi) \in D \subset \mathbb{R}^3_{t,x,\xi}$;

2) *function* $f(t,x,\xi)$ *satisfies the inequality;*

$$\left|f(t,x_1,\xi_1) - f(t,x_2,\xi_2)\right| \le \psi(|x_1 - x_2| + |\xi_1 - \xi_2|),$$

*where* $\psi(x) > 0$ *for* $x \in (0,d]$, $\psi(x)$ *is continuous and monotone nondecreasing function for* $x \in (0,\delta)$, $\delta \in (0,d]$, *and, moreover*

$$\int_{\varepsilon}^{d} \frac{dx}{\psi(x)} \to +\infty \quad for \quad \varepsilon \to 0;$$

3) *operator* $\varphi$ *satisfies the Lipschitz condition* 4) *of Theorem* 1.

*Then Cauchy problem* (1)–(3) *has not more than one solution.*

In [21] analogues of the classical Karatheodory existence theorem, and Perron's, Rosenblatt's uniqueness theorems for differential equations with hysteresis operator are also proved. Other sufficient conditions for existence and/or uniqueness of solutions of differential-hysteresis operator equations, including vector case, are given in previous works of V.I. Borzdyko.

In the paper [23] general conditions of solvability of Cauchy initial value problem for equations with hysteresis are given, and a uniqueness theorem for such equations is proved.

In [24] the questions of existence, continuability and a number of solutions of Cauchy problem for differential equations with Mroz hysteresis operator are considered.

Various aspects of differential equations ordinary as well as partial differential equations are considered in monographs [4, 9, 10], and in the collective extensive survey [20].

### 3.2. Stability and oscillations

The problems of stability and existence of oscillations in systems with hysteresis were studied by many researchers starting from classical works of Andronov and Bautin (1944), Fel'dbaum (1949), Zheleztsov (1959). In these works low-dimensional, a namely, second-order systems were considered. The corresponding results are briefly presented in [25, 26]. For multi-dimensional systems the methods of investigation of stability and existence of oscillations in systems with hysteresis nonlinearities were developed by Yakubovich, beginning with the works [1, 2], and continued in his consequent works [27, 28].

Below we present some results, obtained later. Frequency criteria of absolute stability and existence of oscillatory solutions were obtained. Here we present some results.

Consider a system, described by the equations

$$\begin{cases} \dot{x} = Ax + b\xi(t), \ \sigma = c^* x, \\ \xi(t) = \varphi[\sigma(\tau)|_0^t; \xi_0](t), \end{cases} \tag{4}$$

where $x = x(t)$, $x \in \mathbb{R}^n$, $A$ is a real constant matrix of dimension $n \times n$, $b$ and $c$ are real constant vectors of dimension $n$, and $\varphi[\cdot\,;\xi_0]$ is a hysteresis function (operator). Here the sign denotes transposition.

Let $W(p) = c^*(A - pI)^{-1}b$ be a transfer function of the linear part of system (4). Assume that $A$ is a Hurwitz matrix and hysteresis function $\varphi$ satisfies the fallowing relation

$$0 \le \varphi[\sigma(\tau)|_0^t;\ \xi_0](t)\sigma(t) \le \mu_0 \sigma(t)^2 \quad (\mu_0 \le +\infty). \tag{5}$$

Let us further assume that the *frequency condition* [2]

$$\pi(\omega) \equiv \mu_0^{-1} + Re[(1 + i\omega\theta)W(i\omega)] > 0 \quad \text{for} \quad \omega \in [0,\infty],$$

$$0 < \lim_{\omega \to \infty} \omega^2 \pi(\omega) \le \infty$$

is fulfilled for some number $\theta \ge 0$. Then under some mild additional conditions of the hysteresis function $\varphi[\cdot\,;\xi_0]$ the system (4) is *absolute stable*, i.e.

$$x(t) \to 0 \quad \text{as } t \to \infty, \quad \|x(t)\| \le \psi(\|x(0)\|),$$

for any solution $x(t)$ of the system (4), where $\psi(r)$ is some continuous, nonnegative and nondecreasing function, $\psi(0) = 0$, independent of $\xi_0$ and hysteresis function $\varphi[\cdot\,;\xi_0]$.

In the work [28] a frequency criterion was obtained for stability of the system (4) in another class of hysteresis nonlinearities.

The following theorem improves the Theorem 1 from the work [2].

**Theorem 4** [29, 30]**.** *Let $A$ be a Hurwitz matrix, and let hysteresis function $\varphi[\cdot\,;\xi_0]$ satisfy the relation* (5)*. Assume that transfer function $W(p)$ is nondegenerate. Suppose the following conditions are satisfied:*

1) *system* (4) *is dissipative;*

2) *there exist numbers $\delta > 0$, $\varepsilon > 0$, $\tau \ge 0$ and $\theta$ such that*

$$\frac{\tau}{\mu_0} + Re(\tau + i\omega\theta)W(i\omega) - \varepsilon\omega^2|W(i\omega)|^2 \ge \delta \quad \forall\omega \ge 0;$$

3) *there exists a continuous function $F(\sigma)$ and a number $\nu$ such that*

$$|\varphi[\sigma(\tau)|_0^t;\ \xi_0](t) - F(\sigma(t))| \le \nu|\varphi[\sigma(\tau)|_0^t;\ \xi_0](t)|;$$

4) $4\delta\varepsilon > (\nu\theta)^2$.

*Then $x(t) \to 0$ as $t \to +\infty$ for any solution of system* (4).

*If, in addition to* 1)–4)*, the hysteresis function $\varphi[\cdot\,;\xi_0]$ possesses the property of limit continuity, i.e.*

$$\sigma(t) \to \sigma_*,\ \varphi[\sigma(\tau)|_0^t;\ \xi_0](t) \to \xi_*\ (as\ t \to +\infty) \Rightarrow \varphi[\sigma_*;\xi_*](t) \equiv \xi_*,$$

*then $x(t) \equiv 0$ is trivial solution of system* (4) *and* $\lim\limits_{t\to\infty} x(t) = 0$. *If the function $F(\sigma)$ satisfies the condition* (5)*, then system* (4) *is absolute stable.*

In [30] a frequency condition of *stabilization* of the system (4) by using a harmonic external action is obtained.

Consider a system

$$\begin{cases} \dot{x} = Ax + b[\xi(t) + \alpha\sin(\omega_0 t)], \ \ \sigma = c^* x, \\ \xi(t) = \varphi[\sigma(\tau)|_0^t;\ \xi_0](t), \end{cases} \tag{6}$$

where $x$, $b$, $c$ and $\varphi$ are the same as in system (4); $A$ is a Hurwitz $(n \times n)$-matrix, $\alpha$ and $\omega_0$ are positive numbers.

A solution of system (6) is a pair $(x(t), \xi(t))$, $x(t) \in \mathbb{R}^n$, $\xi(t) \in \mathbb{R}$, where $x(t)$ is an absolute continuous function, and $\xi(t)$ is a continuous one.

Suppose the following hysteresis operator $\varphi$ satisfies the following conditions:

1. The value $\xi(t)$ of the operator $\varphi$ is an absolute function, and there exist constants $\sigma_1, \sigma_2 \in \mathbb{R}$, such that

$$0 \le \frac{d\varphi[\sigma(\tau)|_0^t;\ \xi_0]}{dt} \cdot \frac{d\sigma}{dt} \le \mu\left(\frac{d\sigma}{dt}\right)^2$$

for almost all $t:\ t \notin [\sigma_1, \sigma_2]$;

2. The inequality

$$\left|\frac{d\varphi[\sigma(\tau)|_0^t;\ \xi_0]}{dt} \cdot \frac{d\sigma}{dt}\right| \le m\left(\frac{d\sigma}{dt}\right)^2$$

holds for almost all $t$.

3. Function $\varphi[\sigma(\tau)|_0^t;\ \xi_0]\,(t)$ is bounded:

$$|\varphi[\sigma(\tau)|_0^t;\ \xi_0]\,(t)| \le l$$

for all $\sigma(\tau)$ and $t$. Here $\mu$, $m$ and $l$ are some positive numbers.

Introduce the following notations:

$$\gamma(t) = c^* e^{At} b, \qquad \nu = \int_0^{+\infty} |\gamma(t)| dt, \qquad \nu_1 = \int_0^{+\infty} |\gamma'(t)| dt;$$

$$\rho = -c^* b; \quad T = \frac{1}{\omega_0}[\arcsin \frac{\sigma_2 + \nu l}{\alpha |W(i\omega_0)|} - \arcsin \frac{\sigma_1 - \nu l}{\alpha |W(i\omega_0)|}].$$

We assume that

$$\frac{|\sigma_1 - \nu l|}{\alpha |W(i\omega_0)|} < 1, \qquad \frac{|\sigma_2 + \nu l|}{\alpha |W(i\omega_0)|} < 1.$$

For sufficiently large $\alpha$ these inequalities are satisfied. Further, $\nu < +\infty$, $\nu_1 < +\infty$, since $A$ is a Hurwitz matrix.

**Theorem 5** [30]**.** *Let the pair* $(A, b)$ *be a controllable and* $(A,c)$ *an observable. Suppose the hysteresis operator* $\varphi$ *satisfies the above conditions* 1–3.

*Further, suppose that for some numbers* $\delta_1 > 0$, $\delta_2 > 0$, $\lambda > 0$ *the following conditions are satisfied:*

1) *all poles of transfer function* $W(p - \lambda)$ *have negative real parts;*

2) *the following inequality holds for all* $\omega \geq 0$:

$$\frac{1}{\mu} + Re\, W(i\omega - \lambda)\ \delta_1 |W(i\omega - \lambda)|^2 - \delta_2 |(i\omega - \lambda) W(i\omega - \lambda) - \rho|^2 \geq 0;$$

3) *the inequality*

$$\lambda \frac{2\pi}{\omega_0} > T \frac{|m(1 + m\mu^{-1}) - \delta_1|}{\sqrt{\delta_1 \delta_2}}$$

*is satisfied.*

*Then the relation*

$$\lim_{t \to +\infty} |x_1(t) - x_2(t)| = 0$$

*is fulfilled for any two solutions* $(x_1(t), \xi_1(t))$ *and* $(x_2(t), \xi_2(t))$ *of system* (6).

***Remark.*** Since $A$ is Hurwitz matrix and hysteresis function $\varphi[\sigma(\tau)|_0^t;\ \xi_0]$ is bounded, then the system (6) is dissipative. From here it follows existence of $\frac{2\pi}{\omega_0}$ – periodic solution of system (6) by Browder fixed point theorem. Therefore any solution of system (6) tends to this $\frac{2\pi}{\omega_0}$ – periodic solution. The latter is adequate to "hunting phenomenon" under frequency of external action.

Now we render some *instability* result. Consider a system

$$\begin{cases} \dot{x} = Ax + b\xi(t), \\ \dot{\sigma} = c^* x, \\ \xi(t) = \varphi[\sigma(\tau)|_0^t;\ \xi_0](t), \end{cases} \tag{7}$$

where $A$, $b$ and $c$ are the very same as in (4), $\varphi[\cdot\,; \xi_0]$ is a hysteresis function one of the standard types: play or backlash (Fig. 1), relay with hysteresis (Fig. 2), relay with hysteresis and dead zone (Fig. 3).

Let $K(p)$ be a transfer function of the linear part of the system (7) from input $\xi$ to output $(-\dot{\sigma})$.

The following theorem holds.

**Theorem 6** [26]. *Suppose the following conditions are satisfied:*

1) *function* $p^{-1}K(p)$ *is nondegenerate,* $\lim_{P \to \infty} pK(p) = \Gamma > 0$*;*

2) *there exists a number* $\lambda > 0$ *such that*

$$Re\, K(i\omega - \lambda) < 0 \quad \forall \omega \geq 0; \qquad \lim_{\omega \to \infty} \omega^2 ReK(i\omega - \lambda) < 0;$$

3) *function* $K(p - \lambda)$ *has one positive pole and* $n - 1$ *poles with negative real part;*

4) *the system*

$$\ddot{\eta}(t) + \alpha_0 \dot{\eta}(t) + \zeta(t) = 0, \qquad \zeta(t) = \varphi[\eta(\tau)|_0^t;\ \xi_0](t), \tag{8}$$

*is not stable in the whole for* $\alpha_0 = \dfrac{\lambda}{\sqrt{\Gamma}}$.

*Then the system* (7) *is not stable in the whole.*

The proof of Theorem 6 is based on the Leonov's method of nonlocal reduction, which allows to extend any instability criterion, obtained for (8), on multi-dimensional system of the type (7).

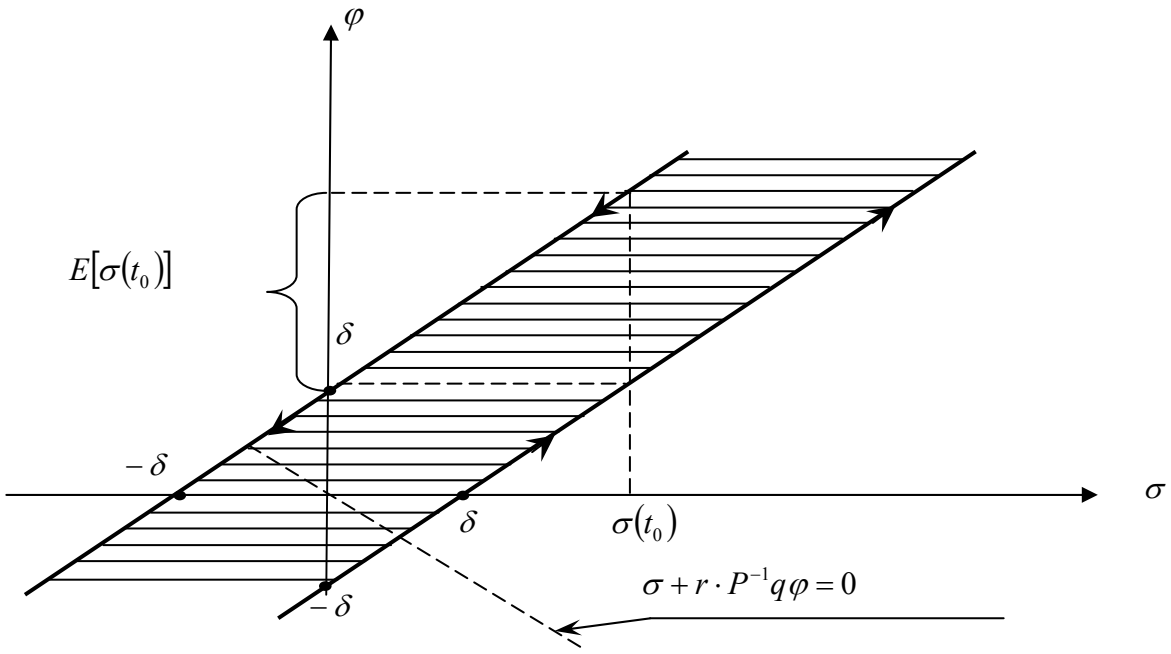


Fig. 1. Hysteresis nonlinearity of the type play

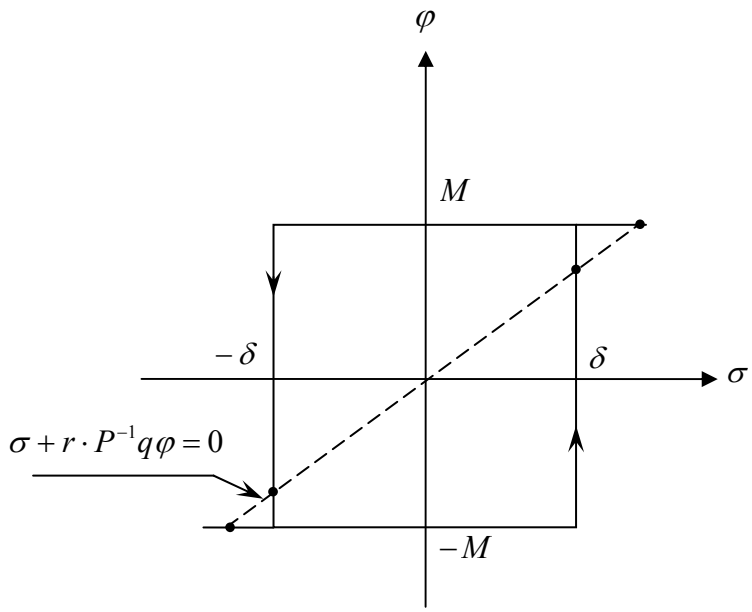


Fig. 2. Relay with hysteresis

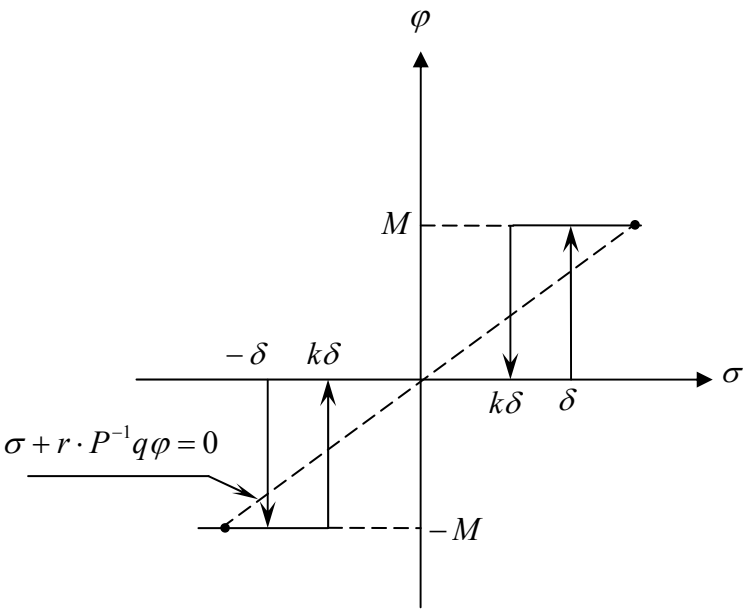


Fig. 3. Relay with hysteresis and dead zone

The following theorem is related to existence of oscillatory solution of the system (7).

**Theorem 7** [26]. *Suppose in the system* (7) *the conditions* 1)–3) *of Theorem* 5 *are satisfied. Let* $A$ *be a Hurwitz matrix and* $c^*A^{-1}b > 0$. *Assume that in the system* (8) *there exists two-sided* $[-\alpha, \beta]$ *– oscillation* $(\alpha > 0,\ \beta > 0)$ *for* $\alpha_0 = \lambda\Gamma^{-0,5}$.

*Then in the system* (7) *there exist also two-sided* $[-\alpha, \beta]$ *– oscillation, i.e. for any number* $T \in [t_0, +\infty)$ *there exist numbers* $t_1, t_2 \in [t_0, +\infty)$ *such that*

$$\sigma(t_1) < -\alpha, \quad \sigma(t_2) > \beta.$$

We also mention Logemann and Ryan work [31], where an existence and regularity theorem is proved for integral equations which contain hysteresis nonlinearities such as play (or backlash), plastic-elastic (or stop) and Prandtl operators. On the basis of this result, frequency stability criteria are derived for feedback system with the hysteresis nonlinearities.

Now we formulate a new theorem for the system (4) with hysteresis operator $\varphi[\cdot\,;\xi_0]$, which is either *Prandtl operator* or *play operator.*

**Theorem 8.** *Assume that matrix* $A$ *has either* $n$eigenvalues *with negative real parts or* $n-1$ *eigenvalues with negative real parts and* $\det A = 0$, *matrix* $A + \theta bc^*$, $\theta \in (0,1]$ *has* $n$ *eigenvalues with negative real parts. Suppose*

$$1 + Re(i\omega) > 0 \qquad \forall\omega \in \mathbb{R}$$

*for Prandtl operator, and*

$$1 + \mu|\mathrm{W}(i\omega)|^2 + (1+\mu)Re\mathrm{W}(i\omega) > 0 \quad \forall\omega \in \mathbb{R}$$

*for modified play operator, where the segments of straight lines, located in hysteresis strip-region (formed by two parallel slanting lines) has slope equal to* $\mu$.

*Then any solution of the system* (4) *tends to stationary set as* $t \to +\infty$.

In conclusion we note that the studies in the field of differential equations with hysteresis operators are intensively continuing. For more information about research in this area, we refer to [4, 9, 10, 20] and the survey [32].

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