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Second-order stochastic differential equations: stability, dissipativity, periodicity. IV. – A survey*

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Abstract. *This paper is a continuation of the previous papers and presents the fourth part of the author's work. The paper reviews results concerning qualitative properties of second-order stochastic differential equations and systems. In the first part we gave a short overview on stability of solutions of the second-order stochastic differential equations and systems by Lyapunov functions techniques and introduced some mathematical preliminaries from probability theory and stochastic processes. In the second part the construction of Ito's and Stratonovich's stochastic integrals is given. In the third part, analog of the chain rule for stochastic differentials (Ito's formula) is presented. The stochastic differential equations in the sense of Ito and in the sense of Stratonovich are introduced. The existence and uniqueness theorem for solutions of stochastic differential equations is formulated. In the present fourth part of the work basic facts from the theory of stability of stochastic differential equations are briefly given. The basic definitions of stability in different senses of stochastic differential systems are presented, the basic general theorems on stability are formulated in terms of the existence of Lyapunov functions, which are stochastic analogs of the classical Lyapunov's theorems on stability. The concept of stochastic dissipative systems is given. A theorem is formulated which gives conditions for existence of periodic and stationary solutions in terms of auxiliary functions for differential equations with a random periodic in time right-hand side, which is a periodic or stationary process.*

Keywords: *random variable, stochastic process, Wiener process, stochastic integral, stochastic differential, Ito formula, stochastic differential equation*

Review article

Стохастические дифференциальные уравнения второго порядка: Устойчивость, диссипативность и периодичность. IV. – Обзор**

(*Рецензирована*)

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Статья представляет собой расширенный текст пленарного доклада на Третьей международной научной конференции «Осенние математические чтения в Адыгее» (ОМЧА – 3), 15–20 октября 2019 г., АГУ, Майкоп, Республика Адыгея.

Аннотация. *Настоящая статья является продолжением предыдущей статьи и представляет собой четвертую часть работы автора. В работе делается обзор результатов исследований качественных свойств решений стохастических дифференциальных уравнений и систем второго порядка. В первой части был дан краткий обзор результатов работ по стохастической устойчивости решений дифференциальных уравнений и систем второго порядка с использованием аппарата функций Ляпунова. Были приведены некоторые предварительные сведения из теории вероятностей и теории случайных процессов. Во второй части дана конструкция стохастических интегралов Ито и Стратоновича. В третьей части дано понятие стохастического дифференциала, приведена формула Ито дифференцирования сложной функции для стохастических дифференциалов, дано определение стохастического дифференциального уравнения в форме Ито и в форме Стратоновича, сформулирована теорема существования и единственности для решений стохастических дифференциальных уравнений. В настоящей, четвертой, части работы даются вкратце основные сведения из теории устойчивости стохастических дифференциальных уравнений Ито. Приводятся основные определения устойчивости в различных смыслах стохастических дифференциальных систем, формулируются основные общие теоремы об устойчивости в терминах существования функций Ляпунова, являющиеся стохастическими аналогами классических теорем Ляпунова об устойчивости. Дается понятие о стохастических диссипативных системах. Приводится теорема, дающая условия существования периодических и стационарных решений в терминах вспомогательных функций для дифференциальных уравнений со случайной периодической по времени правой частью, представляющей собой периодический или стационарный процесс.*

Ключевые слова*: случайная величина, стохастический процесс, винеровский процесс, стохастический интеграл, стохастический дифференциал, формула Ито, стохастическое дифференциальное уравнение*

This paper is a continuation of the previous papers [1–3]. We continue the section "2. Some Mathematical Preliminaries", where some basic notions and facts from probability theory and stochastic analysis, including the stochastic differential equations are introduced. In [3] the notion of stochastic differential is introduced and presented the chain rule, Ito's formula, for stochastic differentials. The definition of stochastic differential equations in the Ito and in the Stratonovich forms is given and formulated the existence and uniqueness theorem for solutions of stochastic differential equations.

Here below we briefly present basic facts from the theory of stability of stochastic differential equations. The basic definitions of stability in different senses of stochastic differential systems are given, the basic general theorems on stability are formulated in terms of the existence of Lyapunov functions, which are stochastic analogs of the classical Lyapunov's theorems on stability. The concept of stochastic dissipative systems is given. A theorem is formulated which gives conditions for existence of periodic and stationary solutions in terms of auxiliary functions for differential equations with a random periodic in time right-hand side, which is a periodic or stationary process.

We keep the general numeration of sections, definitions, theorems and formulas in the work and continue this here.

2.18. Stability of Stochastic Differential Equations

Roughly speaking, the stability means insensitivity of the state of the system to small changes in the initial state or the parameters of the system. The basic facts and methods on the theory of stability of deterministic systems described by ordinary differential equations can be found, for instance, in the books [4–9]. As for the theory of stability of stochastic differential equations, for a detailed account and further references, we refer to the books by Kushner [10], Khasminsky [11], L. Arnold [12], Mao [13]. We will essentially follow the Khasminsky's monograph [11].

2.18.1. Assumptions. Basic Definitions

Consider a stochastic differential equation (in the sense of Ito)

$$dx(t) = b(t, x(t))dt + \sum_{r=1}^{m} \sigma_r(t, x(t))d\xi_r(t) \tag{2.27}$$

with the respect to process $x(t) = x(t, \omega)$, where $x(t)$, $b(t,x)$, $\sigma_r(t,x)$ are vector functions: $x(\bullet)$: $[t_0, \infty) \to \mathbb{R}^n$, $b(\bullet,\bullet)$: $[t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma_r(\bullet,\bullet)$: $[t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\xi_r(t)$ $(r = 1,...,m)$ are independent Wiener processes.

We assume that the assumptions of the existence-and-uniqueness theorem 2.1 (see [3]), taking into account the remark 5 to this theorem, are fulfilled. Also assume that $b(t,x)$ and $\sigma_r(t,x)$ are continuous with respect to t. Furthermore, let us assume that $x_0 = x_0(\omega)$ is with probability 1 a constant. Then for any given initial value $x(t_0) = x_0 \in \mathbb{R}^n$, the equation (2.27) has a unique (up to equivalence) global solution defined on $[t_0, \infty)$, that is denoted by $x(t;\, t_0, x_0) = x(t, \omega\, ;\, t_0, x_0)$: $x(t_0;\, t_0, x_0) = x_0$. The theorem 2.1 also implies that the solution $x(t;\, t_0, x_0; \omega)$ has continuous sample paths (for almost all ω) and provides the finiteness of the moment $\mathrm{E}\|x(t, \omega\, ;\, t_0, x_0)\|^2$.

Assume furthermore that

$$b(t,0) = 0 \quad \text{and} \quad \sigma_r(t,0) = 0 \quad \text{for all} \quad t \ge t_0 .$$

So, the equation (2.27) has the unique solution $x(t) \equiv 0$ corresponding to the initial value $x(t_0) = 0$. This solution is called the *trivial solution*.

Let $S_h = \{x \in \mathbb{R}^n : \|x\| < h\}$, $0 < h \le \infty$. Denote by $C^{1,2}(\mathbb{R}_+ \mathrm{x} S_h, \mathbb{R}_+)$ the family of all nonnegative functions $V(t,x)$ defined on $\mathbb{R}_+ \times S_h$ such that they are continuously differentiable in coordinates x_i of $x = (x_1,...,x_n)$. Define the *differential operator* L *associated with equation* (2.27) by

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{n} b_i(t,x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n}\left(\sigma(t,x)\sigma(t,x)^{\mathrm{T}}\right)_{ij} \cdot \frac{\partial^2}{\partial x_i \partial x_j}, \tag{2.28}$$

where $\sigma(t,x) = (\sigma_1(t,x),...,\sigma_m(t,x))$ is $n \times m$-matrix with column-vectors $\sigma_1(t,x),...,\sigma_m(t,x)$, T denotes transposition.

If $V \in C^{1,2}$ then we have (in symbolic form)

$$LV(t,x) = \frac{\partial V}{\partial t} + \left(b, \frac{\partial}{\partial x}\right)V + \frac{1}{2}\left(\sigma\sigma^{\mathrm{T}}\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)V ,$$

where $b = b(t,x) = (b_1(t,x),...,b_n(t,x))$, $\sigma = \sigma(t,x) = (\sigma_{ir}(t,x))$ $(i = 1,...,n\, ;\, r = 1,...,m)$, $\partial/\partial x = (\partial/\partial x_1,...,\partial/\partial x_n)$, $(*,*)$ denotes the inner product (scalar product).

If $x(t) = x(t,\omega)$ is a solution of the equation (2.27) and $x(t) \in S_h$, then the $\mathrm{v}(t) = V(t, x(t))$ has, in accordance with Ito's theorem (see [3], section 2.16.2), a stochastic differential (see (2.18))

$$d\mathrm{v}(t) = dV(t, x(t)) = LV(t, x(t))dt + \sum_{r=1}^{m}\left(\frac{\partial V(t, x(t))}{\partial x}, \sigma_r(t, x(t))\right)d\xi_r(t).$$

(This explains why the differential operator L is defined as above.)

The latter differential relation can be rewritten in the integral form as

$$V(t, x(t)) - V(t_0, x(t_0)) = \int_{t_0}^{t} LV(s, x(s))ds + \sum_{r=1}^{m}\int_{t_0}^{t}\left(\frac{\partial V(s, x(s))}{\partial x}, \sigma_r(s, x(s))\right)d\xi_r(s).$$

Calculating the expectation of the left-and right-hand sides of the latter and using the properties of stochastic integrals and Fubini's theorem, we get

$$\mathrm{E}\left[V(t, x(t)) - V(t_0, x(t_0))\right] = \int_{t_0}^{t} \mathrm{E}(LV(s, x(s)))ds.$$

(Here we assume that the expectations of the left and the right there exist.)

Now, a stable system should have the property that $\mathrm{v}(t)$ does not increase. In the deterministic case $\sigma(t, x) \equiv 0$ it is required the inequality $\dot{V} = \partial V/\partial t + (b(t, x), \partial V/\partial x) \le 0$. But in the stochastic case, it makes sense to require instead of $\dot{V} \le 0$ the inequality $\mathrm{E}(d\mathrm{v}(t)) \le 0$ (i.e. "$d\mathrm{v}(t) \le 0$ on the average"). Since

$$\mathrm{E}(d\mathrm{v}(t)) = \mathrm{E}(LV(t, x(t))) \quad (\mathrm{E}(d\xi_r(t)) = 0)$$

the requirement $\mathrm{E}(d\mathrm{v}(t)) \le 0$ will be satisfied if

$$LV(t, x) \le 0 \quad \text{for all} \quad t \ge t_0, \quad x \in \mathbb{R}^n.$$

This is the stochastic analog of the requirement that $\dot{V} \le 0$ in the deterministic case and $LV \le 0$ reduces to the $\dot{V} \le 0$ if $\sigma(t, x) \equiv 0$.

Let us know formulate basic definitions of stochastic stability. The concept of stability of the trivial solution $x(t, \omega) \equiv 0$ of stochastic differential equations as in deterministic case can be given in various senses. We shall confine ourselves to those which are popular in the mathematical literature.

Definition 50 ([11, p. 162, 155, 157, 171])**.** 1. *A trivial solution* $x(t, \omega) \equiv 0$ *of equation* (2.27) *is said to be stable in probability or stochastically stable* (*in the strong sense*) *for* $t > 0$ *if for any* $t_0 \ge 0$ *and* $\varepsilon > 0$

$$\lim_{x_0 \to 0} \mathrm{P}\left\{\omega : \sup_{t > t_0}\left\|x(t, \omega; t_0, x_0)\right\| > \varepsilon\right\} = 0.$$

Otherwise, it is said to be stochastically unstable.

2. *The trivial solution* $x(t, \omega) \equiv 0$ *of equation* (2.27) *is said to be stochastically asymptotically stable (in the strong sense) if it is stochastically stable and moreover*

$$\lim_{x_0 \to 0} \mathrm{P}\left\{\omega : \lim_{t \to \infty} x(t, \omega; t_0, x_0) = 0\right\} = 1.$$

3. *The trivial solution* $x(t, \omega) \equiv 0$ *of equation* (2.28) *is said to be stochastically asymptotically stable in the large if it is stochastically stable abd, moreover, for all* $t_0 \ge 0$, $x_0 \in \mathbb{R}^n$,

$$\mathrm{P}\left\{\omega : \lim_{t\to\infty} x(t,\omega\ ;t_0,x_0)=0\right\}=1.$$

4. *The trivial solution* $x(t,\omega)\equiv 0$ *of equation* (2.27) *is said to be exponentially stable in mean square, if for some positive constants* A *and* α

$$\mathrm{E}\left\|x(t,\omega\ ;t_0,x_0)\right\|^2 \le \mathrm{A}\left\|x_0\right\|^2 \exp\left\{-\alpha(t-t_0)\right\} \quad \textit{on} \quad [t_0,\infty) \quad \textit{for all} \quad x_0\in\mathbb{R}^n.$$

To formulate stochastic stability theorems, we need the following definitions (first introduced by A.M. Lyapunov [4]; see also [5–9]).

Definition 51. 1. *A continuous scalar function* $\mathrm{V}(x)$ *defined on a spherical neighborhood of the zero point* $S_h=\left\{x\in\mathbb{R}^n : \|x\|<h\right\}$ $(0<h\le\infty)$ *is said to be positive-definite* (in *the sense of Lyapunov) if* $\mathrm{V}(0)=0$, $\mathrm{V}(x)>0$ *for all* $x\in S_h$, $x\ne 0$.

2. *A continuous scalar function* $\mathrm{V}(t,x)$ *defined on* $[t_0,\infty)\times S_h$ *is said to be positive-definite (in the sense of Lyapunov) if* $\mathrm{V}(t,0)\equiv 0$ *and there exists a positive-definite function* $\mathrm{W}(x)$ *such that* $\mathrm{V}(t,x)\ge \mathrm{W}(x)$ *for all* $(t,x)\in[t_0,\infty)\times S_h$, $x\ne 0$.

3. *A continuous scalar function* V *is said to be negative-definite if* $-\mathrm{V}$ *is positive-definite.*

4. *A continuous non-negative scalar function* $\mathrm{V}(t,x)$ *is said to be decrescent (or is said to have an arbitrary small upper bound or to have an infinitesimal upper limit as* $x\to 0$) *if* $\lim\limits_{x\to 0}\sup\limits_{t\ge t_0}\mathrm{V}(t,x)=0$. (*This relation holds if there exists a positive-definite function* $\mathrm{U}(x)$ *such that* $0\le \mathrm{V}(t,x)\le \mathrm{U}(x)$ *for all* $(t,x)\in[t_0,\infty)\times S_h$.)

5. *A continuous scalar function* $\mathrm{V}(t,x)$ is said to be *radially unbounded* if $\lim\limits_{\|x\|\to\infty}\inf\limits_{t\ge t_0}\mathrm{V}(t,x)=\infty$.

Notice that every positive-definite function $\mathrm{V}(x)$ that is independent of t is decrescent.

2.18.2. Basic general theorems

Suppose the assumptions at the beginning of the subsection 2.18.1 are fulfilled.

Denote by $C_0^{1,2}\left([t_0,\infty)\times S_h\ ,\mathbb{R}_+\right)$ the family of all nonnegative functions $\mathrm{V}(t,x)$ defined on a half-cylinder $[t_0,\infty)\times S_h$ that is everywhere, with the possible exception of the set $\{x=0\}$, continuously differentiable with respect to t and continuously twice differentiable with respect to every component of x. It is clear that the family $C^{1,2}\subset C_0^{1,2}$.

The following theorems are analogous to the well-known theorems of Lyapunov for deterministic systems.

Theorem 2.2 ([11, p. 152]). *Suppose that there exists a positive-definite function* $\mathrm{V}(t,x)\in C_0^{1,2}\left([t_0,\infty)\times S_h\ ,\mathbb{R}_+\right)$ *such that*

$$L\mathrm{V}(t,x)\le 0$$

for all $t\ge t_0$ *and* $0<\|x\|<h$, *where* L *is the differential operator* (2.28).

Then the trivial solution $x(t,\omega)\equiv 0$ *of the equation* (2.27) *is stochastically stable.*

Theorem 2.3 ([11, p. 155]). *Suppose that there exists a positive-definite function* $\mathrm{V}(t,x)\in C_0^{1,2}\left(\left[t_0,\infty\right)\times S_h\ ,\mathbb{R}_+\right)$ *that has an infinitesimal upper limit as* $x\to 0$ (*i.e.* V *is decrescent*) *and* $L\mathrm{V}(t,x)$ *is negative-definite:*

$$L\mathrm{V}(t,x)\le -\mathrm{W}(x)<0 \quad (\mathrm{W}(0)=0)$$

for all $(t,x)\in\left[t_0,\infty\right)\times S_h$.

Then the trivial solution $x(t,\omega)\equiv 0$ *of equation* (2.27) *is stochastically asymptotically stable.*

The following theorem is a generalization of the well-known theorem of Barbashin and Krasovsky [9, p. 248] to stochastic equations.

Theorem 2.4 ([11, p. 158]). *Suppose that there exists a positive-definite function* $\mathrm{V}(t,x)\in C_0^{1,2}\left(\left[t_0,\infty\right)\times\mathbb{R}^n,\mathbb{R}_+\right)$ *that has an infinitesimal upper limit as* $x\to 0$ (*i.e.* V *is decrescent*) *and* $\mathrm{V}(t,x)$ *is radially unbounded function,*

$$\inf_{t\ge t_0}\mathrm{V}(t,x)\to\infty \quad as \quad \|x\|\to\infty,$$

such that $L\mathrm{V}(t,x)$ *is negative-definite.*

Then the trivial solution $x(t,\omega)\equiv 0$ *of equation* (2.27) *is stochastically asymptotically stable in the large.*

Remark. For an autonomous equation

$$dx(t)=b\left(x(t)\right)dt+\sum_{r=1}^{m}\sigma_r\left(x(t)\right)d\xi_r(t),$$

$b(0)=\sigma_r(0)=0$, it is sufficient to consider a function $\mathrm{V}(t,x)\equiv\mathrm{V}(x)$ that is independent of t.

The following theorem gives sufficient conditions for exponential stability in mean square of stochastic systems in terms of Lyapunov functions. It may be viewed as a generalization of well-known theorem for deterministic systems (see [7, section 11]).

Theorem 2.5 ([11, p. 171]). *The trivial solution* $x(t,\omega)\equiv 0$ *of the system* (2.27) *is exponentially stable in mean square for* $t\ge t_0$ *if there exists a function* $\mathrm{V}(t,x)\in C_0^{1,2}\left(\left[t_0,\infty\right)\times\mathbb{R}^n,\mathbb{R}_+\right)$, *such that*

$$k_1\|x\|^2\le\mathrm{V}(t,x)\le k_2\|x\|^2,$$

$$L\mathrm{V}(t,x)\le -k_3\|x\|^2$$

for some positive constants k_1, k_2, k_3.

Remark. In [11, p. 171] the theorem 2.4 is formulated for the system (2.27) to be exponentially stability in p th mean ($p\in\mathrm{N}$; in theorem 2.5 $p=2$).

If the system (2.27) is linear one, i.e.

$$dx(t)=b(t)x(t)dt+\sum_{r=1}^{m}\sigma_r(t)x(t)d\xi_r(t), \tag{2.29}$$

where $b(t)$, $\sigma_r(t)$ are $n\times n$-matrices, the theorem 2.4 can be precised and specified.

Theorem 2.6 ([11, p. 185]). *A necessary condition for exponential stability in mean square of the system* (2.29) *is that for every positive-definite quadratic form* (*in* x) *whose coefficients are continuous bounded functions of time there exists a positive-definite quadratic form* $\mathrm{V}(t,x)$ *such that*

$$L\mathrm{V}=-\mathrm{W}.$$

The same condition, with the phrase "for every …" replaced by "for some …" is also sufficient.

Remark. If matrices $b(x)$ and $\sigma_r(x)$ in (2.29) are constant: $b(x)\equiv b$, $\sigma_r(x)\equiv\sigma_r$, i.e. the system (2.29) is autonomous, then the forms $\mathrm{V}(t,x)$ and $\mathrm{W}(t,x)$ in statement of theorem 2.6 may be replaced by forms $\mathrm{V}(x)$ and $\mathrm{W}(x)$ with constant coefficients [11, p. 185].

2.19. Dissipativity and stability in the weak sense of stochastic differential systems

In this and the two next sections we consider a differential equation of the form [11, p. 10]

$$\frac{dx}{dt}=F(x,t)+\sigma(x,t)\xi(t,\omega), \tag{2.30}$$

where $x=x(t)\in\mathbb{R}^n$, $t\in[t_0,\infty)$, $F(x,t)=(F_1(x,t),...,F_n(x,t))$ is a Borel-measurable function defined for $(x,t)\in\mathbb{R}^n\times[t_0,\infty)$, $\sigma(x,t)$ is a $n\times m$-matrix, $\xi(t,\omega)$ is a separable measurable stochastic process with values in $\mathbb{R}^m$.

We dwell on the dissipativity, stability (in the week sense) and periodicity properties of solutions of the equation (2.30).

Definition 52 ([11, p. 13]). *A stochastic process* $\xi(t,\omega)$ ($t\geq 0$) *is said to be bounded in probability if the random variables* $|\xi(t,\omega)|$ *are bounded in probability uniformly in* t, *i.e.*

$$\sup_{t>0}\mathrm{P}\{\|\xi(t,\omega)\|>R\}\to 0 \quad as \quad R\to\infty.$$

Let $x(t,\omega,\ x_0,t_0)$ *be a solution of* (2.30) *with initial condition* $x(t,\omega,\ x_0,t_0)=x_0(\omega)$.

Definition 53 ([11, p.13]). *The system* (2.30) *will be called dissipative if the random variables* $\|x(t,\omega,\ x_0,t_0)\|$ *are bounded in probability uniformly in* $t\geq t_0$ *whenever* $x_0(\omega)$ *satisfies the relation*

$$\mathrm{P}\{\|x_0(\omega)\|<R_0\}=1.$$

Now let us formulate a general theorem that gives sufficient conditions in terms of Lyapunov functions under which the system (2.30) would be dissipative.

Theorem 2.7 ([11, p. 13]). *Let* $\mathrm{V}(x,t)$ *be a non-negative function, defined on the domain* $D=\mathbb{R}^n\times[0,\infty)$, *which satisfies the global Lipschitz condition with respect to* x:

$$|\mathrm{V}(x_2,t)-\mathrm{V}(x_1,t)|<B\|x_2-x_1\|$$

for all (x_1,t), $(x_2,t)\in D$ *with Lipschitz constant* B.

Suppose $\mathrm{V}(x,t)$ *satisfies the conditions*:

a) $\mathrm{V}_R = \inf\limits_{\mathrm{U}_R^C \times \{t>t_0\}} \mathrm{V}(x,t) \to \infty$ *as* $R \to \infty$, *wher* $\mathrm{U}_R^C = \{x \in \mathbb{R}^n : \|x\| \ge R\}$;

b) $\dfrac{d^\circ \mathrm{V}}{dt} \le -c_1 \mathrm{V}$ ($c_1 = const > 0$), *where* $d^\circ \mathrm{V}/\mathrm{dt}$ *is Lyapunov's operator for deterministic part* $dx/\mathrm{dt} = F(x,t)$ *of the system* (2.30).

Let F *and* σ *satisfy the local Lipschitz conditions*

$$\|F(x_2) - F(x_1)\| < B_R \|x_2 - x_1\|,$$

$$\|\sigma(x_2) - \sigma(x_1)\| < B_R \|x_2 - x_1\|,$$

in the domain $\mathrm{U}_R = \{x \in \mathbb{R}^n : \|x\| < R\}$ *with a Lipschitz constant* B_R *which generally depends on* R.

Assume that σ *also satisfy the condition*

$$\sup_{\mathrm{R}^n \times \{t>t_0\}} \|\sigma(x,t)\| \le c_2 .$$

Then the system (2.30) *is dissipative for every stochastic process* $\xi(t,\omega)$ *such that*

$$\sup_{t>0} \mathrm{E}\|\xi(t,\omega)\| < \infty .$$

Remark. Another dissipativity theorems can be find in [11; see section 1.4].

2.20. Stability in the weak sense

In the section 2.18 we have considered stability in the strong sense for stochastic Ito's differential equations. Here in this section we consider stability in the weak sense for stochastic differential equation of the form (2.30).

Consider the equation (2.30) with $F(0,t) = \sigma(0,t) = 0$ for all t in $[0,\infty)$.

Definition 53 ([11, p. 22])**.** *A solution* $x(t,\omega) \equiv 0$ *of the system* (2.30) *is said to be:*

1. (*Weakly*) *stable in probability* (*for* $t \ge t_0$) *if for every* $\varepsilon > 0$ *and* $\delta > 0$ *there exists and* $r > 0$ *such that* $t \ge t_0$ *and* $\|x_0\| < r$, *then*

$$\mathrm{P}\{\|x(t,\omega;\ t_0,x_0)\| > \varepsilon\} < \delta .$$

2. (*Weakly*) *asymptotically stable in probability if it is stable* (*weakly*) *in probability and, for each* $\varepsilon > 0$ *there exists an* $r = r(\varepsilon)$ *such that for* $t \to \infty$

$$\mathrm{P}\{\|x(t,\omega;\ t_0,x_0)\| > \varepsilon\} \to 0 \ \text{ if } \ \|x_0\| < r .$$

3. *Stable* (*weakly*) *in probability in the large if it is stable* (*weakly*) *in probability and if furthermore for every* x_0, $\varepsilon > 0$ *and* $\delta > 0$ *there exists a* $T = T(x_0,\varepsilon,\delta)$ *such that*

$$\mathrm{P}\{\|x(t,\omega,\ t_0,x_0)\| > \varepsilon\} < \delta$$

for all $t \ge T$.

A similar definition is given for asymptotic stability in probability in the large.

Remark. There are also another types of stability of the trivial solution of (2.30) (see [11, p. 22]).

Now we formulate a theorem that gives effective sufficient conditions for stability in terms of the existence of a Lyapunov function for the shortened system $dx/\mathrm{dt} = F(x,t)$.

Theorem 2.8 ([11, p.24]). *Suppose that for the system* (2.30) *there exists a positive-definite Lyapunov function* $\mathrm{V}(x,t)$, *i.e.*

$$\inf_{t>0,\ \|x\|>r} \mathrm{V}(x,t) = \mathrm{V}_r > 0 \quad for \quad r > 0,$$

satisfying the global Lipschitz condition

$$\left|\mathrm{V}(x_2,t) - \mathrm{V}(x_1,t)\right| \le B\|x_2 - x_1\|$$

for all $(x_1,t),(x_2,t) \in \mathbb{R}^n \times [0,\infty)$, *and conditions*

$$\mathrm{V}(0,t) \equiv 0,$$

$$\frac{d^\circ \mathrm{V}}{dt} \le -c_1\mathrm{V}, \qquad \|\sigma(x,t)\| \le c_2 \mathrm{V}(x,t)$$

($c_1, c_2 > 0$ *are constant*).

Suppose moreover that the process $\|\xi(t,\omega)\|$ *satisfies the law of large numbers: for each* $\varepsilon > 0$, $\delta > 0$ *there exists a* $T > 0$ *such that for all* $t > T$

$$\mathrm{P}\left\{\left|\frac{1}{t}\int_{t_0}^{t_0+t} \|\xi(s,\omega)\| ds - \frac{1}{t}\int_{t_0}^{t_0+t} \mathrm{E}\|\xi(s,\omega)\| ds\right| > \delta\right\} < \varepsilon,$$

and the condition

$$\sup_{t>0} \mathrm{E}\|\xi(t,\omega)\| < \frac{c_1}{Bc_2}.$$

Then the trivial solution $x(t,\omega) \equiv 0$ *of the systems* (2.30) *is asymptotically stable in probability in the large.*

Remark. In [11, p. 25] a theorem of stability in pth mean of trivial solution of (2.30) is also proved.

2.21. Stationary and periodic solutions of stochastic differential equations

An important part of the qualitative theory of stochastic differential equations is the study of existence conditions and properties of periodic and stationary solutions of differential equations whose right-hand side is a periodic or stationary process in t for fixed values of the space variable x. In this section we present a general theorem that allows to give effective sufficient conditions for the existence of stationary and periodic solutions in terms of Lyapunov's type functions.

First we remind the definition of stationary process.

A stochastic process $\xi(t) = \xi(t,\omega)$ $(-\infty < t < \infty)$ with values in $\mathbb{R}^n$ is said to be stationary (*in the restricted sense*) if for every finite sequence of numbers $t_1,...,t_n$ the joint distribution of the random variables $\xi(t_1+h),...,\xi(t_n+h)$ is independent of h.

If we replace the arbitrary number h in the definition of stationary process by a multiple of a fixed number θ, $h = k\theta$ $(k = \pm 1, \pm 2, ...)$, we get the definition of periodic stochastic process with period θ.

A stochastic process $\xi(t)$ is said to be stationary in the wide sense if

$$E\xi(t)=m=const\,,\qquad K(s,t)=K(t-s),$$

where E is an expectation, K is a covariance function

$$K(s,t)=\operatorname{cov}\left(\xi(s),\xi(t)\right).$$

If $\xi(t)$ is a stationary stochastic process with finite expectation and finite variance, then $\xi(t)$ shall also be stationary process in the wide sense. Notice that, for Gaussian processes these both notions coincide. If $\xi(t)$ is a θ-periodic stochastic process, then the functions $\mathrm{E}\xi(t)=m(t)$, $\operatorname{var}\xi(t)=D(t)$ and $K(s,t)$ are periodic with the same period θ:

$$m(t+\theta)=m(t),\quad D(t+\theta)=D(t),$$

$$K(s+\theta,\ t+\theta)=K(s,t).$$

A process that satisfies the latter relations is said to be periodic process in the wide sense.

The following theorem holds.

Theorem 2.9 ([11, p.51])**.** *Suppose that the vector* $F(x,t)$ *and the matrix* $\sigma(x,t)$ *are* θ*-periodic in* t *and that they satisfy a local Lipschitz condition; let further* $F(0,t)$ *be absolutely integrable over every finite interval, and*

$$\sup_{x,t}\left\|\sigma(x,t)\right\|<\infty.$$

Assume moreover that for the shortened system

$$\frac{dx}{dt}=F(x,t),$$

there exists a Lyapunov function $\mathrm{V}(x,t)$ *satisfying the global Lipschitz condition*

$$\left|\mathrm{V}(x_2,t)-\mathrm{V}(x_1,t)\right|<B\cdot\left\|x_2-x_1\right\|\quad (B=const)$$

for all $(x_1,t),(x_2,t)\in\mathbb{R}^n\times[0,\infty)$, *and the following conditions:*

1. $\mathrm{V}(x,t)$ *is nonnegative, and*

$$\inf_{t>0}\mathrm{V}(x,t)\to\infty\quad as\quad \|x\|\to\infty.$$

2. $d^{\circ}\mathrm{V}/dt$ *is bounded above, and*

$$\sup_{t>0}\frac{d^{\circ}\mathrm{V}}{dt}\to-\infty\quad as\quad \|x\|\to\infty.$$

Then the system (2.30) *has a* θ*-periodic solution for each* θ*-periodic stochastically continuous process* $\xi(t,\omega)$ *with finite expectation.*

If F *and* σ *are independent of* t *and* $\xi(t,\omega)$ *is a stationary process, then the same conditions imply the existence of a stationary solution.*

A theorem that gives conditions for convergence to a periodic solution of the system (2.30) is presented and proved in [11, p. 55].

To be continued.

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