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Second-order stochastic differential equations: stability, dissipativity, periodicity. V. – A survey*

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Abstract. *This paper is a continuation of the previous papers and presents the fifth final part of the author's work. The paper surveys the results concerning stability, dissipativity and periodicity properties of the second-order stochastic differential equations and systems. Some new developments in the theory of stability of stochastic differential equations based on the use of the modifying Lyapunov's second method are presented. The work consists of five parts. In the first two parts we have introduced mathematical preliminaries from probability theory and stochastic processes including the construction of Ito and Stratonovich stochastic integrals. In the third part, some facts from the theory of stochastic differential equations are presented. The existence and uniqueness theorems for stochastic systems are formulated. In the fourth part, definitions are provided and basic facts from the theory of stability of stochastic differential equations are given. The basic general Lyapunov-like theorems on stochastic stability, dissipativity and periodicity for solutions of systems considered are formulated in the terms of the existence of Lyapunov functions. Here in the present fifth part, effective sufficient conditions of stability in probability, exponential stability in mean square for the second-order stochastic differential equations and systems are given. Also we give sufficient conditions for dissipativity and periodicity of random processes defined by nonlinear second-order differential equations with random right-hand sides. As an example the harmonic oscillator disturbed by white noise is considered. In the final section of the present paper, we briefly review some new publications related to stochastic stability that characterizes the state - of - the - art of the theory.*

Keywords*: stochastic process, Wiener process, stochastic differential equation, stability in probability, exponential stability in mean square, dissipativity, periodicity, Lyapunov function*

Обзорная статья

Стохастические дифференциальные уравнения второго порядка: Устойчивость, диссипативность и периодичность. V. – Обзор**

(*Рецензирована*)

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Статья представляет собой расширенный текст пленарного доклада на Третьей международной научной конференции «Осенние математические чтения в Адыгее» (ОМЧА – 3), 15–20 октября 2019 г., АГУ, Майкоп, Республика Адыгея.

Аннотация. *Данная статья является продолжением предыдущей и представляет собой пятую, заключительную, часть работы автора. В работе делается обзор результатов исследований, касающихся свойств устойчивости, диссипативности и существования периодических решений стохастических дифференциальных уравнений и систем второго порядка. Приводятся результаты исследований, развивающие теорию устойчивости стохастических дифференциальных уравнений на основе модифицированного второго метода Ляпунова. Работа состоит из пяти частей. В первых двух частях были приведены предварительные сведения из теории вероятностей и случайных процессов, включая построение стохастических интегралов Ито и Стратоновича. В третьей части работы приведены некоторые факты из теории стохастических дифференциальных уравнений. Сформулированы теоремы существования и единственности для стохастических систем. В четвертой части приведены определения и даны основные сведения из теории устойчивости стохастических дифференциальных уравнений Ито. Общие теоремы об устойчивости, диссипативности и периодичности решений рассматриваемых систем сформулированы в терминах существования функций Ляпунова. В настоящей, пятой, части работы даны эффективные достаточные условия устойчивости по вероятности и экспоненциальной устойчивости в среднем квадратическом решений стохастических дифференциальных уравнений и систем второго порядка. Также даны достаточные условия диссипативности и периодичности случайных процессов, определяемых нелинейными дифференциальными уравнениями второго порядка со случайными правыми частями. В качестве примера рассматривается гармонический осциллятор, возмущенный белым шумом. В последнем разделе настоящей статьи сделан краткий обзор работ по стохастической устойчивости, которые характеризуют текущее состояние теории.*

Ключевые слова: *стохастический процесс, винеровский процесс, стохастическое дифференциальное уравнение, устойчивость по вероятности, экспоненциальная устойчивость в среднем квадратическом, диссипативность, периодичность, функция Ляпунова*

The paper is the final part of the five-part author's work and this is a continuation of the four previous papers [1–4]. In [1] the basic notions from probability theory and stochastic analysis were introduced. In [2] the construction of Ito's and Stratonovich's stochastic integrals was given and their basic properties were presented. In [3] some facts from the theory of stochastic differential equations are provided. The stochastic differential equations in the sense of Ito and in the sense of Stratonovich are introduced, and the relation between these two forms of equations is established. The existence and uniqueness theorems for stochastic systems are formulated. In [4] basic facts from the theory of stability of stochastic differential equations are briefly given. The basic general Lyapunov-like theorems on stochastic stability, dissipativity and periodicity for solutions of systems considered are formulated in the terms of the existence of Lyapunov functions. Here in the present fifth part, we will give effective sufficient conditions for stability in probability, exponential stability in mean square of the second-order stochastic differential equations and systems. Also we give sufficient conditions for dissipativity and periodicity of random processes defined by nonlinear second-order differential equations with random right-hand sides. As an example the harmonic oscillator perturbed by white noise is considered. In the final section of the present paper we briefly review new publications related to stochastic stability that characterize the state - of - the - art of the theory.

3. Stability of the Second-order Stochastic Differential Equations and Two-Dimensional Systems

In this section we present some effective sufficient conditions for stability of solutions of the second-order stochastic differential equations.

In what follows we only need to consider the stability of the trivial (zero) solution of systems to be considered. To this case one can reduce stability of an arbitrary solution by introducing new variables, equal to the deviations of the corresponding coordinates of the "perturbed" motion from "unperturbed" one (this procedure is similar to deterministic case).

3.1. Stability of the harmonic oscillator perturbed by white noise

Consider a harmonic oscillator with eigen frequency ω subject to the action of a damping force proportional to velocity with coefficient k. In many problems it seems natural to assume that parameters k and ω are merely the mean value of the damping coefficient and eigen frequency, while their true values are *stochastic* processes with small correlation interval. Such a system can be described by stochastic equation

$$\ddot{x}+\left(k+\sigma_1\dot{\xi}(t)\right)\dot{x}+\left(\omega^2+\sigma_2\dot{\xi}(t)\right)x=0, \tag{3.1}$$

where $x=x(t)$ is a displacement of a material point from the equilibrium position at the time t, $\dot{\xi}(t)$ is a white noise of unity intensity, σ_1^2 and σ_2^2 are the intensities of white noises acting on the frequency and damping coefficient, respectively.

The equation (3.1) can be interpreted in two senses: as an Ito or a Stratonovich stochastic equations. By setting $\dot{x}=y$, one can rewrite the equation (3.1) as a system of Ito's differential equations

$$\begin{cases} dx=ydt, \\ dy=-\left(\omega^2x+ky\right)dt-\left(\sigma_2x+\sigma_1y\right)d\xi(t), \end{cases} \tag{3.2}$$

where $d\xi(t)$ is the stochastic differential of the Wiener process $\xi(t)$ (in the sense of Ito).

By conversion formula (see proposition 22 in [2]) the equation (3.1) interpreted in the Stratonovich form is equivalent to the system of Ito's form

$$\begin{cases} dx=ydt, \\ dy=-\left[\left(\omega^2-\sigma_1\sigma_2/2\right)x+\left(k-\sigma_1^2/2\right)y\right]dt-\left(\sigma_2x+\sigma_1y\right)d\xi(t). \end{cases} \tag{3.3}$$

By using a Lyapunov function in the quadratic form one can prove the following

Proposition 1 ([5]). *The trivial solution* $(x(t)\equiv 0, y(t)\equiv 0)$ *of the system* (3.1) *is exponentially stable in mean square if and only if*:

a) $\sigma_2^2<2k\omega^2$ *in the case* $\sigma_1=0$, $\sigma_2\neq 0$;

b) $\sigma_1^2<2k$ *in the case* $\sigma_1\neq 0$, $\sigma_2=0$;

c) $\sigma^2<2k\omega^2/\left(\omega^2+1\right)$ *in the case* $\sigma_1=\sigma_2=\sigma$.

From proposition 1 and Kushner's theorem [6, p. 39] it follows that for any solution $\left(x(t,\omega),y(t,\omega)\right)$ of the system (3.2) we have $x(t,\omega)\to 0$, $y(t,\omega)\to 0$ as $t\to+\infty$ with probability 1.

For the system (3.3) we get the analogous statement.

Proposition 2 ([5]). *The trivial solution of the system* (3.3) *is exponentially stable in mean square if and only if*:

a) $\sigma_2^2<2k\omega^2$ *in the case* $\sigma_1=0$, $\sigma_2\neq 0$;

b) $\sigma_1^2<k$ *in the case* $\sigma_1\neq 0$, $\sigma_2=0$;

c) $\sigma^2<\omega^2+\dfrac{k+1}{2}-\sqrt{\omega^4+(1-k)\omega^2+(k+1)^2/4}$ *in the case* $\sigma_1=\sigma_2=0$.

In addition, for any solution $\left(x(t),y(t)\right)$ *of the system* (3.3) $x(t)\to 0$, $y(t)\to 0$ *as* $t\to\infty$ *with probability* 1.

Comparing the inequalities a)–c) from propositions 1 and 2 we see that the stability

conditions coincide in the case $\sigma_1 \neq 0$, $\sigma_2 = 0$ for both the Ito's and Stratonovich's form of the equation (3.1), but in the case $\sigma_1 = 0$, $\sigma_2 \neq 0$ the stability condition in the Ito form is weaker than corresponding stability condition for equation (3.1) in the Stratonovich form. In the case $\sigma_1 = \sigma_2$, the condition c) in propositions 1 and 2 are not uniquely comparable.

3.2. Stability of the second-order nonlinear stochastic differential equations

In this subsection we provide some sufficient conditions for stability in probability and exponential stability in the mean square of trivial solution of the second-order nonlinear stochastic differential equations.

Consider a stochastic equation

$$\ddot{x} + f(x)\dot{x} + g(x) + \sigma_1(\dot{x})\dot{\xi}_1(t) + \sigma_2(x)\dot{\xi}_2(t) = 0, \tag{3.4}$$

where the functions $f(x)$, $g(x)$, $\sigma_1(y)$, $\sigma_2(x)$ satisfy the Lipschitz condition, $\dot{\xi}_1(t)$ and $\dot{\xi}_2(t)$ are independent Gaussian white noise processes with unit intensity.

The equation (3.4) can be written as a system of Ito stochastic differential equations

$$\begin{cases} dx = ydt, \\ dy = -\left[yf(x) + g(x)\right]dt - \sigma_1(y)d\xi_1(t) - \sigma_2(x)d\xi_2(t), \end{cases} \tag{3.5}$$

where $d\xi_1(t)$ and $d\xi_2(t)$ are stochastic differentials to be understood in the sense of Ito.

If the equation is interpreted as stochastic differential equation in the sense of Stratonovich, then by conversion formula (proposition 22 in [2]) we can rewrite it in the equivalent Ito's form as

$$\begin{cases} dx = ydt, \\ dy = -\left[yf(x) + g(x) - \dfrac{\sigma_1(y)\sigma_1'(y)}{2}\right]dt - \sigma_1(y)d\xi_1(t) - \sigma_2(x)d\xi_2(t). \end{cases}$$

The system (3.3) is a special case of the system (3.5): if $f(x) = k$, $g(x) = \omega^2 x$, $\sigma_1(\dot{x}) = \sigma_1\dot{x}$, $\sigma_2(x) = \sigma_2 x$ and $\xi_1(t) = \xi_2(t)$, then we get (3.3).

The following theorem holds.

Theorem 3.1 ([7]). *Suppose the following conditions are satisfied* :

a) $b_1 < f(x) < b_2$ *for all* $x \in \mathbb{R}$;

b) $b_3 < g(x)/x < b_4$ *for all* $x \in \mathbb{R}$, $x \neq 0$, $g(0) = 0$;

c) $0 < \sigma_1(y)/y < \beta_1$ *for all* $y \in \mathbb{R}$, $y \neq 0$, $\sigma_1(0) = 0$;

d) $0 < \sigma_2(x)/x < \beta_2$ *for all* $x \in \mathbb{R}$, $x \neq 0$, $\sigma_2(0) = 0$;

e) *there exists a constant* $\alpha > 0$ *such that*

$$b_3 - \alpha\beta_2^2/2 > 0, \quad \alpha\left(b_1 - \beta_1^2/2\right) > 1,$$

where b_1, b_2, b_3, b_4, β_1, β_2 *are some positive constants.*

Then the trivial (zero) solution of the system (3.5) *is exponentially stable in mean square.*

Remark 1. The condition e) is equivalent to the inequality $2b_1b_3 > b_3\beta_1^2 + \beta_2^2$.

Remark 2. In the linear case $f(x) = b_1$, $g(x) = b_3 x$, $\sigma_1(y) = \beta_1 y$, $\sigma_2(x) = \beta_2 x$ the conditions of theorem 3.1 turn into necessary and sufficient ones ([8, p. 222]):

$2b_1b_3 > b_3\beta_1^2 + \beta_2^2$, $b_1 > 0$, $b_3 > 0$.

The proof of the theorem 3.1 is based on considering of special Lyapunov function

$$\mathrm{V}(x,y) = \int_0^x sf(s)\,ds + xy + \frac{\alpha}{2}y^2 + \alpha\int_0^x g(s)\,ds \tag{3.6}$$

and subsequent application of the theorem 2.5 (see [4]).

In the particular case $f(x) = b$, $g(x) = 2cx + 3x^2$, $\alpha = 2$ the function (3.6) turns to the function constructed in [9] for the system (3.5), where $f(x) = b$, $g(x) = 2cx + 3x^2$, $\sigma_1(y) = 0$, $\sigma_2(x) = \sigma x$.

Consider a special case for the system (3.5): $\sigma_1(y) = \sigma_1 y$, $\sigma_2(x) = \sigma_2 x$. Then using the Lyapunov function

$$\mathrm{V} = \left(2b - \sigma_1^2\right)\int_0^x sf(s)\,ds + 2\int_0^x g(s)\,ds + \left(2b - \sigma_1^2\right)xy + y^2$$

by virtue of theorem 2.2 ([4]) one can establish the following statement.

Theorem 3.2 ([7]). *Assume that in the system* (3.5) $\sigma_1(y) = \sigma_1 y$, $\sigma_2(x) = \sigma_2 x$. *If the following conditions are satisfied* :

a) $f(x) > b > 0$ *for all* $|x| < \varepsilon$, $\varepsilon > 0$;

b) $\left(2b - \sigma_1^2\right)g(x)/x - \sigma_2^2 \ge 0$ *for all* $|x| < \varepsilon$, $x \ne 0$;

c) $2b - \sigma_1^2 > 0$,

then the trivial solution of the system (3.5) *is stochastically stable.*

The theorem 3.2 is a generalization of a statement presented in [9] for the particular case $f(x) = b$, $g(x) = 2cx + 3x^2$, $\sigma_1(y) = \sigma_1 y$, $\sigma_2(x) = 0$.

Consider a stochastic equation

$$\ddot{x} + f(x,\dot{x})\dot{x} + g(x) + \sigma x\dot{\xi}(t) = 0 \quad (\sigma = const). \tag{3.7}$$

The equation (3.7) is to be interpreted in the sense of Ito or in the sense of Stratonovich.

Using the Lyapunov function

$$\mathrm{V} = \frac{b_1}{2}x^2 + xy + \frac{\alpha}{2}y^2 + \alpha\int_0^x g(s)\,ds$$

and applying the theorem 2.4 ([4]) one can prove the following

Theorem 3.3 ([7]). *Suppose that* :

a) $0 < b_1 < f(x,\dot{x}) < b_1 + 2\sqrt{(1-\alpha b_1)\left(\alpha\sigma^2/2 - b_2\right)}$ *for all* x, $\dot{x}$;

b) $g(x)/x > b_2 > 0$ *for all* $x \ne 0$, $g(0) = 0$;

c) *there exists a constant* $\alpha > 0$ *such that* $\alpha b_1 > 1$ *and* $b_2 > \alpha\sigma^2/2$.

Then the trivial solution $x = 0$, $\dot{x} = 0$ *of the equation* (3.7) *is stochastically asymptotically stable in the large.*

Consider the stochastic equation

$$\ddot{x} + F(\dot{x}) + g(x) + \sigma\dot{x}\dot{\xi}(t) = 0. \tag{3.8}$$

The following theorem holds.

Theorem 3.4 ([7]). *Suppose that* :

a) $b_1+\varepsilon<F(y)/y<b_1+2b_2-\varepsilon$ *for all* $y\neq 0$, $F(0)=0$;

b) $b_2<g(x)/x<b_3$ *for all* $x\neq 0$, $g(0)=0$;

c) $\sigma^2<2b_1$,

where b_1, b_2, b_3, ε *are some positive constants. Then the trivial solution* $x=0$, $\dot{x}=0$ *of the equation* (3.8) *is exponentially stable in mean square.*

The proof of theorem 3.4 is carried out using the special Lyapunov function

$$\mathrm{V}=(b_2+1)\left(y^2+2\int_0^x g(s)\,ds\right)+\left(2b_1-\sigma^2\right)\left(\frac{b_1^2}{2}x^2+xy\right)$$

and applying the general theorem 2.5([4]).

In the linear case $F(\dot{x})=b_1\dot{x}$, $g(x)=b_2x$ the conditions of theorem 3.4 turn to necessary and sufficient conditions ([8, p. 222]): $\sigma^2<2b_1$, $b_2>0$.

Analogous stochastic stability conditions one can derived for the equations (3.4), (3.7) and (3.8) if they are interpreted in the sense of Stratonovich. For this it should convert the stochastic systems in the Stratonovich form into stochastic systems in the Ito form by using proposition 22 in [2].

In [10] the nonlinear stochastic differential equation of the form

$$\ddot{x}+b(x,\dot{x},t)+g(x)=\sigma(x,\dot{x},t)\dot{\xi}(t)+\varepsilon(t)$$

is considered to be a generalization of above equations. Here $\varepsilon(t)$ is an external driving force. For latter equation an asymptotic bound is established.

Now we consider perturbations from another class of random processes different from white noises. Here we present a theorem of stability in probability for the stochastic equation

$$\ddot{x}+f(x)\dot{x}+g(x)=\sigma(x,\dot{x})\xi(t,\omega)$$

or for the equivalent system

$$\dot{x}=y,\quad \dot{y}=-f(x)y-g(x)+\sigma(x,y)\xi(t,\omega), \tag{3.9}$$

where $f(x)$, $g(x)$ and $\sigma(x,y)$ satisfy a local Lipschitz condition, $\xi(t,\omega)$ is a measurable random process with values in $\mathbb{R}$ and almost surely (i.e. with probability 1) integrable over every finite interval.

Assume that $f(0)=g(0)=\sigma(0,0)=0$. This implies that ($x(t,\omega)\equiv 0$, $y(t,\omega)\equiv 0$) is trivial solution of (3.9).

Introduce a Lyapunov type function

$$\mathrm{V}(x,y)=\left(y^2/2+\gamma xy+\int_0^x g(s)\,ds+\beta x^2/2\right)^{1/2}, \tag{3.10}$$

where β and γ are some real numbers which are chosen later.

We set

$$\mathrm{B}=\sup_{(x_i,y_i)\in\mathbb{R}^2}\frac{\left|\mathrm{V}(x_2,y_2)-\mathrm{V}(x_1,y_1)\right|}{|x_2-x_1|+|y_2-y_1|}.$$

Suppose the stochastic process $\xi(t,\omega)$ satisfies the law of large numbers.

The following theorem holds.

Theorem 3.5 ([11]). *Suppose the functions* $f(x)$, $g(x)$ *and* $\sigma(x,y)$ *satisfy the conditions* :

a) $b_1 < f(x) < b_2$ *for all* x;

b) $b_3 < g(x)/x < b_4$ *for all* $x \neq 0$;

c) $|\sigma(x,y)| \le b_5 (x^2+y^2)^{1/2}$ *for all* $(x,y) \in \mathbb{R}^2$;

d) $\sup_{t>0} \mathrm{E}|\xi(t,\omega)| < b_6/b_5 \mathrm{B}$,

where b_i $(i=1,...,6)$ *are some positive numbers*,

e) *the numbers* β *and* γ *in* (3.10) *are such that* $\gamma^2 < \beta < 4b_1b_3/b_2^2$.

Then the trivial solution of the system (3.9) *is asymptotically stable* (*weakly*) *in the large*.

The proof of theorem 3.5 is based on using the Lyapunov function (3.10) and subsequent application of general theorem 2.8 ([4]).

3.3. Stability of the two-dimensional autonomous stochastic systems

Here we present some sufficient conditions for stochastic stability of trivial solution of the second-order systems of stochastic differential equations of the first order.

3.3.1. Linear stochastic systems

We start by considering linear autonomous homogeneous system whose parameters are perturbed by white noise

$$\begin{cases} \dot{x} = \left(a+\sigma_1\dot{\xi}(t)\right)x + \left(b+\sigma_2\dot{\xi}(t)\right)y, \\ \dot{y} = \left(c+\sigma_3\dot{\xi}(t)\right)x + \left(m+\sigma_4\dot{\xi}(t)\right)y, \end{cases} \tag{3.11}$$

where $x = x(t,\omega)$, $y = y(t,\omega)$, $t \in [0,\infty)$, $\dot{\xi}(t)$ is a Gaussian white noise with unit intensity, a, b, c, m, σ_i $(i=1,...,4)$ are real numbers.

As above in subsections 3.1 and 3.2 the system (3.11) can be interpreted in two different senses: as a system of two stochastic differential equations of the Ito's form or Stratonovich's form. Since the Stratonovich's form can be reduced by conversion formula (proposition 22 in [2]) to the Ito's form, we will focus on interpreting the system (3.11) in the Ito's sense:

$$\begin{cases} dx(t) = (ax+by)\,dt + (\sigma_1 x + \sigma_2 y)\,d\xi(t), \\ dy(t) = (cx+my)\,dt + (\sigma_3 x + \sigma_4 y)\,d\xi(t). \end{cases} \tag{3.12}$$

Here $\xi(t) = \xi(t,\omega)$ is a Wiener process, $dx(t)$, $dy(t)$, $d\xi(t)$ are Ito's stochastic differentials.

The Stratonovich form of system (3.11) can be obtained from (3.12) by replacing the coefficients a, b, c and m as follows

$$a \to a + \left(\sigma_1^2 + \sigma_2\sigma_3\right)/2, \quad b \to b + \sigma_2(\sigma_1+\sigma_4)/2,$$

$$c \to c + \sigma_3(\sigma_1+\sigma_4)/2, \quad m \to m + \left(\sigma_4^2 + \sigma_2\sigma_3\right)/2.$$

Let one of the coefficients of the "random" part of the system (3.12) be nonzero (for instance, $\sigma_1 \neq 0$) and others be zero.

Theorem 3.6 ([5]). *Let* $\sigma_1 \neq 0$, $\sigma_2 = \sigma_3 = \sigma_4 = 0$. *Then the trivial solution* $\left(x(t) \equiv 0, y(t) \equiv 0\right)$ *of the system* (3.12) *is stable in probability if*:

1) $a + m < 0$, $am - bc > 0$, $b \neq 0$;

2) $\sigma_1^2 < \dfrac{2(a+m)(bc-am)}{m^2 + (am-bc)}$.

In addition, any function $\left(x(t), y(t)\right)$ *of system* (3.12) *possesses the property*: $\left(x(t), y(t)\right) \to \left\{(x, y) : x = 0\right\}$ *as* $t \to \infty$ *with probability* 1.

The inequality 2) in the theorem 3.6 gives the lower estimate of the bifurcation value of white noise intensity. The theorem 3.6 is proved by using the Lyapunov function

$$\mathrm{V} = (am - bc)x^2 + (mx - by)^2$$

and applying theorem 2.2 ([4]).

The mean square exponential stability conditions of trivial solution is provided by the following

Theorem 3.7 ([5]). *Let* $\sigma_1 \neq 0$, $\sigma_2 = \sigma_3 = \sigma_4 = 0$. *Then for the trivial solution of system* (3.12) *to be exponentially stable in mean square it is necessary and sufficient that the conditions should satisfy* :

1) $a + m < 0$, $am - bc > 0$;

2) $\sigma_1^2 < \min\left\{ \dfrac{2(a+m)(bc-am)}{m^2 + (am-bc)}, \dfrac{p + \sqrt{p^2 + c^2 q}}{c^2} \right\}$, *if* $c \neq 0$, *where*

$$p = (a+m)\left(c^2 + m^2 + am - bc\right) - 2c(ac + bm), \qquad q = 4(am - bc)\left[(a+m)^2 + (b-c)^2\right],$$

and $\sigma_1^2 < -2a$ $(a < 0)$, *if* $c = 0$.

In addition, for any solution $\left(x(t), y(t)\right)$ *of* (3.12) $x(t) \to 0$, $y(t) \to 0$ *as* $t \to \infty$ *with probability* 1.

The inequality in the condition 2) in theorem 3.7 gives the lower bound of the bifurcation value of white noise intensity Σ_0^2 up to which the perturbed system (3.12) remains exponentially stable, and for which the system first becomes unstable.

In the case when two or more coefficients of the "random" part of system (3.12) are different from zero the stability conditions become cumbersome and difficult to use. For the details see [5].

3.3.2. Stability of two-dimensional nonlinear stochastic systems

Now we will consider nonlinear second-order stochastic systems. We provide sufficient conditions for stochastic stability of two-dimensional diffusion process described by a system of nonlinear Ito stochastic differential equations.

Consider a system of two stochastic differential equations of the form

$$\begin{cases} dx_1(t) = \left[f_{11}(x_1) + f_{12}(x_2)\right]dt + \sigma(x_1)\,d\xi(t), \\ dx_2(t) = \left[f_{21}(x_1) + f_{22}(x_2)\right]dt, \end{cases} \tag{3.13}$$

where two of the functions $h_{ij}(x_j) = f_{ij}(x_j)/x_j$ $(x_j \neq 0)$, $i, j = 1, 2$, are constant, $\xi(t)$ is a Wiener process, and the $dx_1(t)$, $dx_2(t)$, $d\xi(t)$ are treated as Ito's stochastic differentials.

It is assumed that the functions $f_{ij}(x_j)$ are continuously differentiable for all x_j, the function $\sigma(x_1)$ satisfies the Lipschitz condition, $f_{ij}(0)=0$, $\sigma(0)=0$. We suppose that two of the four functions f_{ij} $(i, j=1, 2)$ are nonlinear ones, and the other two ones are linear.

The deterministic case $\sigma(x_1)\equiv 0$ was considered by Erugin [12–14] and Malkin [15] for systems of the form (3.13) with a single non-linear function and by Krasovsky [16–18] in the case of two nonlinearities.

Let us formulate some basic results concerning stability of trivial solution of system (3.13).

We will limit ourselves to considering (3.13) in the case:

a) $f_{11}(x_1)=f(x_1)$, $f_{22}(x_2)=g(x_2)$, $f_{12}(x_1)=ax_2$, $f_{21}(x_2)=bx_1$;

b) $f_{11}(x_1)=f(x_1)$, $f_{21}(x_1)=g(x_1)$, $f_{12}(x_1)=ax_2$, $f_{22}(x_2)=bx_2$,

where a and b are real constants.

In the remaining cases of (3.13) the corresponding results are similarly formulated.

In each particular case of the system (3.13) we will use notation without indices.

Consider the stochastic system

$$dx(t)=\left(f(x)+ay\right)dt+\sigma(x)d\xi(t), \qquad dy(t)=\left(bx+g(y)\right)dt, \tag{3.14}$$

where a and b are real constants.

Theorem 3.8 ([19]). *Suppose there exist positive numbers* δ_i $(i=0, 1, 2, 3, 4)$ *and a number* Δ *such that*:

1) $c_1 f(x)/x-ab>\delta_0$ *for all* $x\neq 0$;

2) $f(x)/x+c_1<-\delta_1$ *for all* $x\neq 0$;

3) $c_2 g(y)/y-ab>\delta_2$ *for all* $y\neq 0$;

4) $g(y)/y+c_2<-\delta_3$ *for all* $y\neq 0$;

5) $c_1 f'(x)-ab<\Delta$ *for all* $x\in\mathbb{R}$;

6) $0<\sigma(x)/x<\delta_4$ *for all* $x\neq 0$;

7) $\delta_4^2<2\delta_0\delta_1/\left(c_1^2+\Delta\right)$.

Here c_1 *and* c_2 *are constants such that* $c_1c_2=ab$ *and* $ab\neq 0$.

Then the trivial solution of the system (3.14) *is asymptotically stable in the large.*

If in addition to conditions 1)–7) *the inequalities hold*:

8) $c_1 f(x)/x-ab<\delta_5$ *for all* $x\neq 0$;

9) $c_2 g(y)/y-ab<\delta_6$ *for all* $y\neq 0$,

where δ_5 *and* δ_6 *are some positive numbers, then the trivial solution of system* (3.14) *is exponentially stable in mean square.*

It is easily to check that for the special deterministic and linear case $\sigma(x)\equiv 0$, $f(x)=f_0x$ and $g(y)=g_0y$ $(f_0, g_0\in\mathbb{R})$ the conditions of theorem 3.8 turn into the well-known necessary and sufficient Routh-Hurwitz conditions for the stability of solutions of second-order linear systems with constant coefficients: $f_0+g_0<0$, $f_0g_0-ab>0$. Note that the deterministic part $(\sigma(x)\equiv 0)$ of the system (3.14) is a control system which was investi-

gated by Ayzerman [20].

We deliver another stability theorem for a system of the form (3.13).

Consider the system

$$\begin{cases} dx(t) = \left(f(x) + ay\right)dt + \sigma(x)\,d\xi(t), \\ dy(t) = \left(g(x) + by\right)dt, \end{cases} \tag{3.15}$$

where a and b are constant.

Theorem 3.9 ([19]). *Suppose there exist positive numbers* δ_0, δ_1, δ_2 *and a number* Δ *such that the following conditions are satisfied* :

1) $f(x)/x + b + \varepsilon < -\delta_0$ *for all* $x \neq 0$;

2) $(b+\varepsilon) f(x)/x - a\, g(x)/x > \delta_1$ *for all* $x \neq 0$;

3) $(b+\varepsilon) f'(x) - ag'(x) < \Delta$ *for all* $x \in \mathbb{R}$;

4) $0 < \sigma(x)/x < \delta_2$ *for all* $x \neq 0$;

5) $\delta_2^2 < 2\delta_0\delta_1 / \left(b^2 + \Delta\right)$.

Here $\varepsilon > 0$ *is a sufficiently small number.*

Then the trivial solution of the system (3.15) *is asymptotically stable in the large.*

If in addition to conditions 1)–5) *the inequality*

$$(b+\varepsilon) f(x)/x - a\, g(x)/x < \delta_3 \quad \text{for all} \quad x \neq 0 \quad (\delta_3 > 0)$$

hold, then the trivial solution of system (3.15) *is exponentially stable in mean square.*

As above in the deterministic linear case the conditions of theorem 3.9 turn into the Routh-Hurwitz stability conditions.

Let us apply theorem 3.9 to the system

$$dx(t) = \left(y - \varphi(x)\right)dt + \sigma(x)\,d\xi(t), \qquad dy(t) = -h(x)\,dt, \tag{3.16}$$

whose deterministic part ($\sigma(x) \equiv 0$) is equivalent to the classical Lienard equation

$$\ddot{x} + \varphi'(x)\dot{x} + h(x) = 0.$$

The system (3.16) is a special case of (3.15), where $f(x) = -\varphi(x)$, $g(x) = -h(x)$, $a = 1$, $b = 0$.

We have

Corollary. *Suppose the following conditions be satisfied for all* $x \in \mathbb{R}$:

1) $x\varphi(x) > (\delta_0 + \varepsilon)x^2$;

2) $x\left(h(x) - \varepsilon\varphi(x)\right) > \delta_1 x^2$;

3) $h'(x) - \varepsilon\varphi'(x) < \Delta$;

4) $0 < x\sigma(x) < \delta_2 x^2$,

and the inequality $\delta_2^2 < 2\delta_0\delta_1/\Delta$ *holds, where* δ_0, δ_1, δ_2 *and* Δ *are some positive numbers.*

Then the trivial solution of the system (3.16) *is asymptotically stable in the large.*

If in addition to above conditions the inequality

$$x\left(h(x) - \varepsilon\varphi(x)\right) < \delta_3 x^2 \quad (\delta_3 > 0)$$

holds for all x, *then the trivial solution of system* (3.16) *is exponentially stable in mean square.*

The proofs of the theorems 3.8 and 3.9 are based on the construction special Lyapunov functions and studying the properties of these functions along the solutions of the systems considered, and on the application of the general theorems 2.4 and 2.5 ([4]).

Assertions similar to Theorems 3.8 and 3.9 also hold for systems of the form (3.13) with diffusion coefficient depending on the coordinate x_2.

4. Dissipativity and Periodicity of Second-Order Stochastic Differential Equations

In [21, 22] sufficient conditions under which the second-order nonlinear differential equations perturbed by a random process possess the dissipativity property are obtained. Also sufficient conditions for existence of stationary and periodic solutions of second-order differential equations with stationary and periodic random right-hand sides are given.

Consider the stochastic equation of the form

$$\ddot{x}+\varphi(x,\dot{x},t)+g(x)=\sigma(x,\dot{x})\xi(t,\omega), \tag{3.17}$$

or the equivalent system

$$\dot{x}=y, \quad \dot{y}=-\varphi(x,\dot{x},t)-g(x)+\sigma(x,\dot{x})\xi(t,\omega), \tag{3.18}$$

where the functions $\varphi(x,y,t)$, $g(x)$ and $\sigma(x,y)$ are such that the system (3.17) satisfy the existence and uniqueness theorems 1.5 and 1.6 from [8, p. 9, 10]. Therefore the system (3.17) with initial conditions $x(t_0)=x_0(\omega)$, $y(t_0)=y_0(\omega)$ determines a new stochastic process which is almost surely absolutely continuous for all $t\ge t_0$.

We note that the deterministic case ($\sigma(x,\dot{x})\equiv 0$) of the equation (3.17), where

1) $\varphi(x,\dot{x},t)\equiv f(x)\dot{x}$; 2) $\varphi(x,\dot{x},t)\equiv F(\dot{x})$; 3) $\varphi(x,\dot{x},t)=f(x,\dot{x})\dot{x}$,

was studied by many authors whose results are presented in detail in the classical monograph [23].

Here we will only state dissipativity and periodicity theorems for the two special cases:

1) $\varphi(x,y,t)\equiv f(x,y)y$ and 2) $\varphi(x,y,t)\equiv F(y)$.

Theorem 4.1 ([21, 22]). *Let* $\varphi(x,y,t)=f(x,y)y$. *Assume that the process* $|\xi(t,\omega)|$ *has a bounded expectation* :

$$\sup_{t\ge t_0}\mathrm{E}|\xi(t,\omega)|<\infty$$

and $|\sigma(x,y)|$ *is bounded* : $|\sigma(x,y)|\le \mathrm{B}$ ($\mathrm{B}=const$) *for all* x, y.

Suppose there exist positive numbers c_1, c_2, c_3, c_4 *and* R *such that* :

1) $c_1<f(x,y)<c_2$ *for all* $(x,y)\in U_{\mathrm{R}}^c$;

where $U_{\mathrm{R}}^c=\{(x,y):x^2+y^2\le \mathrm{R}\}$;

2) $c_3<g(x)/x<c_4$ *for all* $|x|>\mathrm{X}_0$,

where X_0 *is some positive number.*

Then the system (3.17) *is dissipative.*

Note that in [21, 22] the above theorem is formulated under general assumption with respect to function $f(x,y)$.

Theorem 4.2 ([21, 22]). *Let* $\varphi(x,y,t)\equiv F(y)$. *Assume that the process* $|\xi(t,\omega)|$

has a bounded expectation and $|\sigma(x,y)|$ *is bounded.*

Suppose there exist positive numbers c_1, c_2, c_3, c_4, X *and* Y_0 *such that*:

1) $c_1 < g(x)/x < c_2$ *for all* $|x| > \mathrm{X}_0$;

2) $c_3 < F(y)/y < c_3$ *for all* $|y| > \mathrm{Y}_0$.

Then the system (3.17) *is dissipative.*

By considering various narrower classes of random processes $\xi(t,\omega)$ ([8, p. 16]) one can derive various dissipativity conditions under less stringent restrictions on the functions $g(x)$, $f(x,y)$, $F(y)$ and $\sigma(x,y)$.

The following two theorems provide sufficient conditions for the existence of periodic and stationary solutions of (3.17).

Theorem 4.3 ([21, 22])**.** *Suppose the process* $\xi(t,\omega)$ *is a stationary process. Then the conditions of theorems* 4.1 *and* 4.2 *imply the existence for a stationary solution of* (3.17).

Theorem 4.4 ([21, 22])**.** *Let the functions* $f(x,y)$, $F(y)$, $g(x)$ *and* $\sigma(x,y)$ *be replaced by* $f(x,y,t)$, $F(y,t)$, $g(x,t)$ *and* $\sigma(x,y,t)$, *respectively.*

Suppose $f(x,y,t)$, $F(y,t)$, $g(x,t)$ *and* $\sigma(x,y,t)$ *are* T*-periodic in* t *and* $f(0,0,t)$, $F(0,t)$, $g(0,t)$ *and* $\sigma(0,0,t)$ *are absolutely integrable over every finite interval, and* $\sigma(x,y,t)$ *is bounded*:

$$|\sigma(x,y,t)| < \mathrm{B} \quad \textit{for all} \quad x,\ y,\ t \quad (\mathrm{B} = const).$$

Then the conditions of theorem 4.1 *and* 4.2 *provide the existence of a* T*-periodic solution of the system* (3.17) *for each* T*-periodic stochastically continuous process* $\xi(t,\omega)$ *with finite expectation.*

The proofs of the theorems 4.1–4.4 are based on the construction special Lyapunov functions and on the application of general theorems 1.8 and 2.6 from [8, p. 13, 51].

5. A brief Review of Stability of Stochastic Differential Equations

In this section we will present a short overview of some further investigations related to the stability and dissipativity properties of stochastic differential equations.

First we notice that the theory of stochastic stability began to form much later than other sections of the stability theory. This theory arose for the purpose of studying the stabilization of controlled motion in systems perturbed by random noise. The early stages of development of the theory of stochastic stability are in detail presented in the survey of Kozin [24].

The significant influence on the development of the stochastic stability was made by the pioneering works of Bertram and Sarachik [25] and Kac and Krasovsky [26]. In [25] sufficient conditions for stability in the mean square are obtained by an extension of "Lyapunov's Second Method" to stochastic problems. In [26] the authors investigate the stability of the zero solution of the equation $\dot{x} = f(x,t,y(t))$, where $y(t)$ is a time-homogeneous finite-state Markov chain. The solution of this problem is given in terms of Lyapunov functions $\mathrm{V}(x,t)$, in this case instead of the derivative $\dot{\mathrm{V}}(x,t) \equiv d\mathrm{V}/dt$ along the sample path, roughly speaking, the expectation $L\mathrm{V}$ $(\sim \mathrm{E}\dot{\mathrm{V}}(x,t))$ of this derivative is considered. This paper also contains important results concerning the stability of linear systems and stability in the first approximation. The paper by Kac and Krasovsky has stimulated considerable fur-

ther research.

A fairly complete understanding of the approaches of the early period to the analysis of stability of random processes described by ordinary differential equations of the form

$$\dot{x} = f(x,t) + \sigma(x,t)\xi(t,\omega), \quad x \in \mathbb{R}^n, \tag{5.1}$$

where $\xi(t,\omega)$ is a random process from a sufficiently wide class of processes (e.g., the expectation $|\xi(t,\omega)|$ to be bounded) is given in the survey [24] and monograph [8, ch. 1]. The properties of solutions of the systems of the type (5.1) are studied by using Lyapunov functions of the truncated system $\dot{x} = f(x,t)$.

With regard to the stochastic differential equations of the type

$$\dot{x} = f(x,t) + \sigma(x,t)\dot{\xi}(t,\omega), \quad x \in \mathbb{R}^n, \tag{5.2}$$

where $\dot{\xi}(t,\omega)$ is a "while" noise (i.e. a Gaussian process such that $\mathrm{E}\dot{\xi}(t,\omega) = 0$, $\mathrm{E}\left[\xi(s)\cdot\xi(t)\right] = \delta(t-s)$), there exist numereous publications devoted to the stability property of such equations and related topics. For a detailed account and further references we refer to the books by Kushner [6], L. Arnold [27], Friedman [28], Mao [29], Levakov [30], and above all, the profound work by Khasminsky [8] (see also bibliography in these books).

We also mention the monographs by Kolmanovskii and Nosov [31], and Mohammad [32] in which the Lyapunov's second method was developed to deal with the stability of stochastic functional differential equations. In these books a Lyapunov-like theory for the stability of stochastic differential equations has been developed. A sufficiently comprehensive presentation of results on the stability of stochastic differential equations of the type (5.2) by method of Lyapunov functions (Lyapunov's second or direct method) is also given in survey papers [10, 33–36].

Below we present briefly some recent results concerning stability of solutions for stochastic differential equations and related issues.

To begin with, there is a series of works by Friedman and Pinsky [37, 38] and Pinsky [39]. In these papers there are given stability conditions for a point and for an invariant set. In [37] the authors investigate asymptotic behavior of solutions of linear time-independent Ito equations. A sufficient condition for asymptotic stability of the zero solution is given. In the two-dimensional case conditions for spiralling at a linear rate are determined. In [38] sufficient conditions under which the process determined by a system of Ito equations converges to the boundary, that consists on points and surfaces, when $t \to \infty$, are given. In the case of plane domains, the authors give conditions to ensure that the process "spirals". In [39] the author construct asymptotic expansions for the exponential growth rate (Lyapunov exponent) and rotation number of the random oscillator when the noise is small and is defined by a temporally homogeneous Markov process with a finite number of states.

A generalization of the invariance theorem of La Salle (La Salle invariance theorem [40, 41]) to stochastic systems was given by Kushner [42, 43] and Mao [44].

In [45] it is considered a non-linear system of stochastic differential Ito equations of the type

$$dx = \left(Ax + b\varphi(t,\sigma)\right)dt + b\kappa d\mathrm{w}(t), \quad \sigma = c^{\mathrm{T}}x, \quad \kappa = r^{\mathrm{T}}x, \tag{5.3}$$

where $x \in \mathbb{R}^n$, A, b, c, r are real matrices of dimensions $n \times n$, $n \times 1$, $n \times k$, respectively; $\mathrm{w}(t)$ is a k-dimensional Wiener process ($d\mathrm{w}$ is Ito stochastic differential), $\varphi(t,\sigma)$ is a scalar function which satisfies the conditions to ensure that solutions of (5.3) ex-

ists through any point and are unique, and $\varphi(t,0)\equiv 0$. It is assumed that A is Hurwitz matrix. For the systems of the type (5.3) a frequency domain criterion for absolute stochastic stability in the class of nonlinearities

$$\Phi_{\mu_0}=\{\varphi\equiv\varphi(t,\sigma):0\le\varphi/\sigma\le\mu_0\},\quad 0<\mu_0\le+\infty,$$

is derived.

In the works [46, 47] sufficient conditions for asymptotic and exponential stability in mean square of nonlinear stochastic control systems are obtained in frequency domain. The results are derived by using the stochastic analogs of the Lyapunov's second method and the Kalman-Yakubovich-Popov frequency lemma.

In series of works [48–50] algebraic coefficient criteria for stability of solutions of linear systems of Ito stochastic differential equations are obtained. In [48] coefficient criterion and sufficient conditions for asymptotic stability with probability one of solutions of the continuous-time systems of Ito linear parametric stochastic differential equations are derived by the method of Lyapunov functions. These conditions are expressed in terms of the coefficients of the equations: systems' matrix A and parametric random perturbation matrix B. In [49] matrix criterion and sufficient conditions for asymptotic stability and boundedness with probability one of solution of discrete-time linear stationary stochastic difference equations are obtained. In [50] algebraic coefficient criteria and sufficient algebraic conditions for asymptotic stability of solutions of linear continuous-time stochastic time-delayed difference equations are derived by method of Lyapunov function. The criteria and conditions are formulated in terms of the existence of solutions of some matrix equations. In [51] sufficient criteria for asymptotic stability and boundedness of solutions of stochastic differential equations are established via multiple Lyapunov functions.

In 2001 L. Arnold and Schmalfuss ([52]) developed Lyapunov's second method (the method of Lyapunov functions) for random dynamical systems [53] and random sets together with matching notions of attraction and stability. In this case, the Lyapunov functions for random dynamical systems are also random. The authors introduce definitions of stability, attractor and Lyapunov function in such a way that would allow them to prove that a random set is asymptotically stable if and only if it has a Lyapunov function.

In [54] sufficient and necessary conditions for the existence of a stochastically bounded solution of a nonlinear nonhomogeneous stochastic differential equation are obtained. The conditions is given in terms of the exponentially p-stable for some $p>0$ for the solution of the corresponding linearized homogeneous stochastic differential equation. In [55] the boundedness and exponential asymptotic stability of solutions of nonlinear stochastic differential systems are studied. Sufficient conditions for boundedness and exponential asymptotic stability are derived using Lyapunov functions.

In [56] the authors generalize the well-known Barbashin-Krasovsky theorem to the case of stochastic differential equations. They obtain the criteria for asymptotic stability in probability of the zero solution of an autonomous system of stochastic differential equations in the case when $L\mathrm{V}\le 0$ and the set $\mathrm{M}=\{L\mathrm{V}=0\}$ does not include entire semitrajectories of the system considered almost surely. Here $\mathrm{V}=\mathrm{V}(x)$, $x\in\mathbb{R}^n$, is a Lyapunov function, and L is the generator differential operator associated with the system.

In [57] the authors proposed a specific linear combination of subsystems' energies as Lyapunov function for multi-degree-of-freedom nonlinear stochastic dynamical systems, and the corresponding sufficient condition for the asymptotic stability with probability one is derived. In [58] so called stochastic bounded stability property for a general class of nonlinear stochastic systems is studied. The bounded stability property is such that for a given bounded region and realization probability, sample paths remain bounded in the assigned region with

the assigned probability. The authors provide a sufficient condition for the proposed bounded stability to be satisfied conditions based on a Lyapunov-like-function.

In [59] a general class of the nonlinear time-varying systems of Ito stochastic differential equations is considered. Two problems on the partial stability in probability are studied: 1) the stability with respect to a given part of the variables of the zero equilibrium, 2) the stability with respect to a given part of the variables of the "partial" (zero) equilibrium. The stochastic Lyapunov function-based conditions for the partial stability in probability are established. In [60] necessary and sufficient conditions for stability in probability of nontrivial solution of a stochastic nonlinear control system are provided.

Much attention has been given to the investigation of stability problems for systems of stochastic differential equations with delay. Kushner [61], Kolmanovskii [62], Kolmanovskii and Nosov [31, ch. 4], Kolmanovskii and Shaikhet [63], Shaikhet [64] and others proved general theorems of the Lyapunov type. In [65] sufficient conditions for the stability and boundedness of solutions of a stochastic delay differential equations of the second order are obtained by constructing a Lyapunov functional.

The problems of dissipativity for stochastic dynamical systems defined by stochastic differential equations are studied by many authors [66–69] (see also bibliography in them). In these works the notion of dissipativity for stochastic systems considered is understood as a stochastic version of the concept of dissipation in deterministic dynamical systems (with input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^l$, and state $x \in \mathbb{R}^n$) first introduced by Willems [70]. The latter notion (for deterministic systems) involves the real-valued functions $r(u, y)$ (called the *supply rate*), $\mathrm{V}(x)$ (called the *storage function*) that should satisfy some integral inequality, called the *dissipativity inequality*. The papers [71–75] are devoted to the existence of random periodic solutions for stochastic differential equations.

6. Conclusion

The paper consisting of five parts surveys the publications concerned with qualitative properties of the second order linear and nonlinear stochastic differential equations and stochastic systems. In the final fifth part, in the section 5, of the paper a brief overview of stability and related to it tasks of stochastic differential equations is given. Some new developments in the stability theory of stochastic differential equations, based on the use of the method of Lyapunov functions are presented.

The qualitative properties that we adress are stochastic stability, stochastic dissipativity as well as existence of stationary and periodic solutions. We stated theorems which give sufficient conditions for the stochastic systems considered to posses these properties. The results formulated was obtained using Lyapunov's second method. The conditions for stability, dissipativity and periodicity are given in terms of the systems' coefficients.

In the first two parts [1, 2] of the paper some preliminaries from stochastic calculus are given, including the constructions of the Ito's and Stratonovich's stochastic integrals, and their basic properties and the relationship between the two types of the integrals are presented. In the third part [3] some basic facts from the theory of stochastic differential equations are provided. The stochastic differential equations in the sense of Ito and in the sense of Stratonovich are introduced, the relation between the two forms of equations is established, the existence and uniqueness theorems are formulated. The forth part [4] briefly provides basic facts from the theory of stability of stochastic differential equations. The basic general Lyapunov-like theorems on stochastic stability, stochastic dissipativity and periodicity are formulated in terms of the existence of Lyapunov functions.

In the present, fifth, part effective sufficient conditions for stability in probability, exponential stability in mean square of the second-order linear and nonlinear Ito stochastic dif-

ferential equations and systems are given. Also sufficient conditions for dissipativity of random processes defined by nonlinear second-order differential equations with random right-hand sides are provided. For the latter class of stochastic equations sufficient conditions for the existence of periodic and stationary solutions are presented. Necessary and sufficient conditions for stability in mean square of the harmonic oscillator perturbed by white noise are established. The final section of the paper briefly reviews new results and publications related to stability of stochastic differential equations, and characterizes the state-of-the-art of the theory.

Finally, we remark that the problems of the stochastic stability and related to them ones are still active area of research and the flow of publications in this topics does not decrease.

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